and explain why equations (3.28) and (3.29) are to be preferred? Suggestion: Apply the two-level model [in the form of equations (2.5) and (3.2)] to this situation. What are the conditions that \( \frac{n_z}{n_e} - \left( \frac{n_z}{n_e} \right)^*? \)

4. IONIZATION AND RECOMBINATION RATES

When plasma conditions change rapidly, as, for example, in a time-varying stationary plasma or in the rapid expansion of a plasma through a nozzle, departures from the Saha equation can result from finite ionization and recombination rates. In this section we discuss the application of the equations for \( n_k \) and \( n_e \) to the study of such rates.

In general, ionization and recombination phenomena are described by the continuity equations (2.1) and (2.2) for \( n_k \) and \( n_e \), with \( \dot{n}_k \) and \( \dot{n}_e \) given by the rate equations (2.24) and (2.25). To make the situation more manageable, we impose two simplifications on these equations. First, we assume as in Sec. 3 that the pth excited level and all higher levels are in Saha equilibrium with the free electrons, so that equation (3.22) holds for these levels. The rate equations (2.24) and (2.25) for \( \dot{n}_k \) and \( \dot{n}_e \) are then replaced by equations (3.27) and (3.28) or (3.29). The appropriate value for \( p \) in a given situation can be found approximately from the inequality (3.24) or a posteriori from the solution of the rate equations for \( n_k \). In the latter case, a value of \( p \) is chosen, the rate equations are then solved for \( n_k \), and the values of \( n_k \) are examined to verify that they satisfactorily approach Saha-equilibrium values as the pth level is approached from below.

The second simplification is based on the observation that the time for adjustment, or relaxation, of the excited-level populations (\( k > 1 \)) is in many cases much more rapid than the ionization or recombination time. Accordingly, the values of \( n_k \) for \( k > 1 \) will tend to be in balance with the instantaneous values of \( n_e \) and \( n_1 \), and we can set

\[ n_k \approx 0, \quad k > 1. \tag{4.1} \]

This approximation can be made plausible in particular situations by means of order-of-magnitude estimates, based on equations (3.27) and (3.29), of the characteristic times \( \tau_e \) and \( \tau_k \) for adjustment of \( n_e \) and \( n_k \). For example, if \( n_e \) and \( n_k \) are established mainly by collisional processes and if we estimate \( \tau_e \) and \( \tau_k \) from the rates of population, we have

\[ \tau_e \sim \left( \frac{n_e}{n_k} \right)^{-1} \sim \sum_{i < p} n_{i1} S_i \]

and

\[ \tau_k \sim \left( \frac{\dot{n}_k}{n_k} \right)^{-1} \sim \frac{n_k}{n_e \sum_{j < k} n_j S_j + \sum_{p > 1 > j < k} n_j S_j + n_e^2 \sum_{k} S_k} \]
The ratio of the characteristic times is then

$$
\tau_k \sim \frac{n_k}{n_e} \left[ \frac{\sum_{l \leq p} n_l S^l}{\sum_{j < k} n_j S^j + \sum_{p > l \geq k} n_l S_k + n_e^2 \lambda^k S_k} \right].
$$

Under most conditions the factor in brackets in equation (4.2) is less than unity. That this is physically reasonable can be seen from the following argument: In both the numerator and the denominator of this ratio there are terms that are proportional to each of the densities \( n_l \) for \( 1 \leq l < p \) (as well as an additional term which is proportional to \( n_e^2 \) in the denominator). If the coefficient of each of these terms in the denominator is at least as large as the corresponding term in the numerator, the ratio in question will be less than unity. This will be the case if \( S^k \geq S^l \) and \( S_k \geq S^l \). Physically, these inequalities will usually hold, since in a collision between an electron and an excited atom with \( l < p \) the probabilities that the atom will be excited to a higher level or de-excited to a lower level are usually greater than the probability that the atom will be ionized. From equation (4.2) we then find that

$$
\tau_k \ll \tau_e \quad \text{if} \quad n_k \ll n_e, \quad (4.3)
$$
as is usually the case for \( k > 1 \).

With these simplifications we can use equation (3.29) for \( \dot{n}_e \), and the relevant equations become

$$
\frac{\partial n_e}{\partial t} + \nabla \cdot [n_e (u + U_e)] = \dot{n}_e, \quad (4.4)
$$

where

$$
\dot{n}_e = \sum_{k < p} \left[ n_e n_k S^k - n_e^2 (n_e \lambda^k S_k + A_{\lambda \lambda} \beta_{\lambda \lambda}) \right], \quad (4.5)
$$

and

$$
\dot{n}_k = 0 = n_e \sum_{j < k} n_j S^j
$$

$$
- n_k \left[ \sum_{j < k} (n_e \lambda^j S_j + A_{\lambda \lambda} \beta_{\lambda \lambda}) + n_e \left( \sum_{p > l \geq k} \lambda^l S^l + \lambda^k S_k \right) \right]
+ \sum_{p > l \geq k} n_l (n_e \lambda^l S_k + A_{\lambda \lambda} \beta_{\lambda \lambda}) + n_e^2 (n_e \lambda^k S_k + A_{\lambda \lambda} \beta_{\lambda \lambda}) \quad \text{if} \quad (1 < k < p). \quad (4.6)
$$

As in equations (3.26), the symbol \( \lambda \) denotes the set composed of the continuum and the \( p \)th and higher levels, which are assumed to be in Saha
equilibrium with the free electrons. The detailed-balancing relations between the rate coefficients are, from equations (2.15) and (3.26),

\[ \frac{j^S_{k}}{j^S_{j}} = \left( \frac{n_{k}}{n_{j}} \right)^{*} \]  

and

\[ \frac{j^{S'}_{k}}{j^{S'}_{j}} = \left( \frac{n_{k}^{2}}{n_{j}} \right)^{*}. \]  

As before, we regard the equation for \( n_{1} \) as redundant. We note, however, that it follows from \( \dot{n}_{k} = 0 \) \( (k > 1) \) that \( \dot{n}_{1} = -\dot{n}_{e} \), so that an equivalent procedure would be to omit equation (4.5) for \( \dot{n}_{e} \) and retain the equation for \( \dot{n}_{1} \).

The general form of the solution to the foregoing equations can be seen as follows: Assuming that the coefficients \( A_{kj}, \beta_{kj}, j^{S}(T_{e}), \) and \( j^{S'}(T_{e}) \) are known, the homogeneous algebraic equations (4.6) can be solved for the various values \( n_{k}(1 < k < p) \) as functions of \( n_{e}, n_{1}, \) and \( T_{e} \). With the use of these values, equation (4.5) then yields an expression \( \dot{n}_{e} = \dot{n}_{e}(n_{e}, n_{1}, T_{e}) \). It is this expression, when used with equation (4.4), that describes the ionization and recombination process.

As in Sec. 3, we now consider formally the expressions for \( \dot{n}_{e} \) in the limits of high and low electron density. In the corona limit of low electron density, depopulation of the various levels and of \( n_{e} \) occurs primarily as a result of radiative processes rather than by collisional processes. To the extent, then, that \( \dot{n}_{e} \) is established by a balance between collisional ionization from the ground level and two-body radiative recombination [cf. equation (3.21b)] we have directly

\[ \dot{n}_{e} = n_{e} n_{1} j^{S'} - n_{e}^{2} \sum_{k<p} A_{j,k} \beta_{j,k} \quad \text{(corona limit)} \]  

This result is independent of the solution of equation (4.6) for \( n_{k} \), the values of which are given by equation (3.21a) as before. Again we must emphasize, however, that the results for the corona limit such as equation (4.8) may be inaccurate when applied to actual plasmas, since at low electron densities, additional effects such as atom-atom excitation and departures from a Maxwellian distribution for the free electrons may be important. Furthermore, irrespective of these effects, Bates and Kingston (1964) have found that very low electron densities are required in order that the recombination rate [corresponding to the negative term in equation (4.8)] approach its value for purely radiative recombination. Thus, for example, they find that for a hydrogen plasma at \( T_{e} = 250^\circ \text{K} \) collisional effects increase the recombination rate by more than a factor of two for values of \( n_{e} \) greater than about \( 2 \times 10^{5} \text{cm}^{-3} \).
In the limit of high electron density, collisional processes are dominant in comparison with radiative processes, and the recombination mechanism is one of three-body recombination. In this limit, in accord with the inequalities (3.20), the radiative terms can be omitted in the rate equations (4.5) and (4.6), which then become

\[ \dot{n}_e = \sum_{k<P} (n_e n_k kS^k - n_e^{3-k} S_k) \]  

(4.9)

and

\[ \sum_{1<j<k} n_j S^k - n_k \left( \sum_{j<k} S_j + \sum_{p>l>k} kS^l + kS^k \right) + \sum_{p>l>k} n_l S_k \]

\[ = -(n_{1,1} S^k + n_{2,2}^{3-k} S_k). \quad (k > 1) \]  

(4.10)

The densities \( n_k \) for \( 1 < k < p \) can be found from the rate equations (4.10) in terms of \( n_e, n_1, \) and \( T_e \). We have written the terms in \( n_1 \) and \( n_e \) on the right-hand side of equation (4.10) to emphasize the fact that \( n_1 \) and \( n_e \) are regarded here as parameters. With the use of Cramer's rule and the properties of determinants, it can be seen by inspection that the solution to equations (4.10) must be of the form

\[ n_k = a_k n_1 + a_k n_e^2. \]  

(4.11)

The quantities \( a_k \) and \( a_k \) depend on the electron temperature through the rate coefficients \( jS^k \) and \( kS^k \), but not the densities, and can be expressed as ratios of determinants. Furthermore, since radiative processes are omitted from the present calculations, whenever \( n_e^2/n_1 = (n_e/n_1)^* \) we must have \( n_e/n_1 = (n_e/n_1)^* \) [see equation (2.12b)]. It follows that

\[ a_k = \left( \frac{n_k}{n_1} \right)^* - a_k \left( \frac{n_e^2}{n_1} \right)^*. \]

Equation (4.11) then becomes

\[ \frac{n_k}{n_1} = \left( \frac{n_k}{n_1} \right)^* + a_k \left[ \frac{n_e^2}{n_1} - \left( \frac{n_e}{n_1} \right)^* \right]. \]  

(4.12)

We can now use this result in equation (4.9) to obtain for \( \dot{n}_e \)

\[ \dot{n}_e = n_e n_1 \sum_{k<P} \left[ \left( \frac{n_k}{n_1} \right)^* - a_k \left( \frac{n_e^2}{n_1} \right)^* \right] kS^k \]

\[ - n_e^2 \sum_{k<P} \left( ^2 S_k - a_k S^k \right). \]
Applying equation (4.7b), we have finally

\[ \dot{n}_e = \alpha \left[ n_1 \left( \frac{n_e^2}{n_1} \right)^* - n_e^2 \right], \]  

(4.13)

where the overall recombination coefficient \( \alpha \) is given in terms of the quantities \( a_k \) of equation (4.12) by

\[ \alpha = n_e \sum_{k < p} \left( \frac{n_k}{n_1} \right)^* - a_k \right] k S_k^{*}, \]  

(4.14)

It can be seen that in accord with macroscopic reasoning the expression (4.13) for \( \dot{n}_e \) consists of one term which is proportional to \( n_e n_1 \) and another which is proportional to \( -n_e^3 \). Furthermore, equation (4.13) yields the steady-state solution \( n_e/n_1 = (n_e/n_1)^* \) when \( \dot{n}_e = 0 \). This recovery of Saha equilibrium at the steady state is a consequence of the fact that radiative processes have been omitted. We note that for the present collision-dominated situation the linear dependence of \( \alpha \) on \( n_e \), shown by equations (4.14), could have been incorporated explicitly in equation (4.13) by the use of a new coefficient \( \alpha/n_e \). The present form of equation (4.13), however, is consistent with the conventional definition of the recombination coefficient.

Although the evaluation of the recombination coefficient \( \alpha \) from equation (4.14) and the solution of equations (4.10) for \( a_k \) is straightforward, a simpler formulation of three-body recombination would clearly be useful in practical applications. Such a solution has been obtained by means of approximate arguments by Hinnov and Hirschberg (1962). They assume that there exists a certain “critical level,” say the \( q \)th level, which is such that the probability that an electron which recombines into a higher level will subsequently be de-excited to the ground level is equal to the probability that an electron which recombines into a lower level will subsequently be reionized. That is, since \( -\dot{n}_e = \dot{n}_1 \), Hinnov and Hirschberg focus attention on whether an electron which recombines into any level will be reionized before it is de-excited to the ground level. The \( q \)th level is such that the “loss” to the recombination rate of electrons which recombine into higher levels but are reionized before reaching the ground level is just compensated by electrons which reach the ground level via recombination into higher levels. The net recombination rate is then equal to the rate at which electrons recombine into the \( q \)th level and all lower levels, that is

\[ n_e^2 \alpha = n_e^2 \sum_{1 \leq k \leq q} k S_k. \]

(4.15)
We note that the same result would follow from our previous analysis [equations (4.14)] if we took

\[ a_k = \begin{cases} 
0, & 1 \leq k \leq q \\
\frac{n_k}{n_1^2}, & k > q
\end{cases} \]

and replaced \( \lambda' \) by \( \lambda \). When used in equation (4.12), these values of \( a_k \) would then yield

\[ \frac{n_k}{n_1} = \left( \frac{n_k}{n_1} \right)^* \quad \text{for} \quad 1 \leq k \leq q \]

and

\[ \frac{n_k}{n_2} = \left( \frac{n_k}{n_2} \right)^* \quad \text{for} \quad k > q. \] (4.16)

That is, the recombination rate in equation (4.13) would equal that of Hinnov and Hirschberg if it were assumed that the \( q \)th and all lower levels are (precisely) in equilibrium with the ground level and that all higher levels are (precisely) in equilibrium with the free electrons (see Exercise 4.3). We note, however, that this is only a sufficient condition for such agreement and that equations (4.16) differ in detail from the physical basis of the Hinnov and Hirschberg argument.

Hinnov and Hirschberg select the critical \( q \)th level by the criterion that

\[ q \leq q = q - 1 \leq q \]

where the indicated approximation applies to high-lying levels. By equation (4.7a) or (2.15), the equality \( q \leq q = q - 1 \leq q \) will be satisfied when

\[ \frac{g_q}{g_{q-1}} e^{-q - 1/kT_e} = 1. \] (4.17)

Thus, the \( q \)th level is that level where the upward collisional rate \( n_e n_q S_q \) is approximately equal to the downward rate \( n_e n_q S_{q-1} \). For \( k < q \) the downward rate will generally be the greater, whereas for \( k > q \) the upward rate will be greater. Assuming a hydrogenic structure for the high-lying levels of the excited atom [see equation (II 4.8)], equation (4.17) becomes

\[ \left( \frac{2N_q}{2(N_q - 1)} \right)^2 \exp \left\{ -13.6 \left( \frac{1}{kT_e (N_q - 1)^2} - \frac{1}{N_q^2} \right) \right\} = 1, \]
Section 4 Ionization and Recombination Rates 463

where \( N_q \) is the principal quantum number of the \( q \)th level and \( kT_e \) is expressed in electron volts. For \( N_q \gg 1 \), this yields (see Exercise 4.2)

\[
N_q^2 \approx \frac{13.6}{kT_e}
\]

and therefore

\[
\epsilon_{q\lambda} = \frac{13.6}{N_q^2} \approx kT_e. \tag{4.18}
\]

With the \( q \)th level now specified, Hinnov and Hirschberg evaluate \( \alpha S^4 \) from equation (2.9) by use of the Thomson ionization cross section (see Chapter V, Sec. 8)

\[
Q_{kA} = \left( \frac{\epsilon}{4\pi\epsilon_0} \right)^2 \frac{\pi}{\epsilon} \left( \epsilon_{k\lambda} - \frac{1}{\epsilon} \right). \tag{4.19}
\]

The rate integrals \( \alpha S \) which appear in the expression (4.15) for the recombination coefficient \( \alpha \) can then be found from the values \( \alpha S^4 \) by means of the detailed-balancing relation \( \alpha S_{\lambda}/\alpha S^4 = (n_0/n_e^2)^4 \). With the use of certain numerical approximations consistent with \( N_q \gg 1 \) and the hydrogenic approximation for the structure for the excited atom, Hinnov and Hirschberg then calculate from equation (4.15) the recombination coefficient \( \alpha \) in equation (4.13). For \( T_e \geq 3000^\circ \text{K} \) they find that \( \alpha \) is given approximately by the widely quoted formula

\[
\alpha \approx 5.6 \times 10^{-27} n_e (kT_e)^{-9/2} \text{ cm}^3/\text{sec}, \tag{4.20a}
\]

where \( kT_e \) is again expressed in electron volts and \( n_e \) is in cm\(^{-3}\). If \( T_e \) is expressed in \( ^\circ \text{K} \), equation (4.20a) becomes

\[
\alpha \approx 3.4 \times 10^{-22} n_e \left( \frac{T_e}{1000} \right)^{-9/2} \text{ cm}^3/\text{sec}. \tag{4.20b}
\]

When used in the rate equation (4.13) this form for \( \alpha \) yields a relatively simple, although approximate, expression for the term \( \dot{n}_e \) in the electron continuity equation (4.4).

In the calculations of Hinnov and Hirschberg, the levels in the vicinity of the \( q \)th level make the major contribution to the sum (4.15). This suggests that our previous restrictions (3.20) on the relative magnitudes of radiative and collisional processes and, also, considerations of the validity of the hydrogenic approximation need be applied only to the high-lying levels in the vicinity of the \( q \)th level. This situation is indicative of the existence of a relatively small group of levels, such that transitions between these levels limit the rate of the recombination process. Calculations
based on a rate-limited model of this kind have been performed, for example, by Byron, et al. (1963, 1962) and by Dugan (1964).

When both radiative and collisional recombination processes are important, the situation is more complicated. To the extent that radiative escape causes departures from the Saha equation in the steady state, when \( \dot{n}_e = 0 \), it is clear that a macroscopic detailed-balancing relation does not apply to \( \dot{n}_e \). Accordingly, the limiting, steady-state form for \( \dot{n}_e \) must be modified from that of equation (4.13). For this case of coupled radiative and collisional processes, \( \dot{n}_e \) can be evaluated from equations (4.5) and (4.6). Detailed calculations, based on essentially the same formulation, have been performed by Bates, Kingston, and McWhirter (1962). We shall now summarize certain of the results of these calculations.

Bates, Kingston, and McWhirter (1962) have solved numerically the rate equations for \( \dot{n}_e \) and \( \dot{n}_i \) for a plasma which consists of electrons and hydrogen ions. Although they allow for multiply charged ions, for simplicity, we shall restrict our discussion as before to the case where the ionic charge is unity. They consider two situations, that of optically thin plasmas for which all radiation escapes and \( P_{ij} \) and \( P_{ji} \) = 1, and that of plasmas which are optically thick for radiation in the Lyman series, that is, resonance transitions which terminate in the ground level, so that \( P_{kl} = 0 \). In the latter case they consider the following four possibilities for the other radiative processes:

\[
P_{ij} = 1, \quad P_{ik} = 1; \quad P_{ij} = 1, \quad P_{ij} = 0, \quad P_{ij} = 1; \quad P_{ij} = 0, \quad P_{ij} = 0, \quad P_{ij} = 1\text{—where here } j > 1, k \geq 1.
\]

As we have already noted, the formulation of Bates, Kingston, and McWhirter is equivalent to our equations (4.5) and (4.6) for \( \dot{n}_e \) and \( \dot{n}_{k>1} = 0 \). They, however, chose to work with the rate equation for \( \dot{n}_i \) rather than the rate equation for \( \dot{n}_e \). Since \( \dot{n}_i = -\dot{n}_e \) when \( \dot{n}_{k>1} = 0 \), the equation for \( \dot{n}_i \) can be derived from our equations (4.5) and (4.6). Thus we have (see Exercise 4.4)

\[
\dot{n}_i = -\dot{n}_e = -n_1 n_e \left( \sum_{1 \leq i < p} S_{il} + S_{il'} \right) + \sum_{1 \leq i < p} n_i (n_e S_{i1} + A_{i1} ) \\
+ n_2^2 (n_e S_{11} + A_{11} ) .
\] (4.21)

As a result of the fact that equations (4.6) and (4.21) are linear in the ground-level density \( n_1 \), the solution to these equations can be written in the form

\[
\dot{n}_i = -\dot{n}_e = n_2^2 \gamma = n_2^2 (\alpha - S n_1 n_e) .
\] (4.22)

Here \( \gamma, \alpha, \) and \( S \) are defined by these relations, which follow the notation of Bates, et al. The quantities \( \alpha \) and \( S \) depend on \( n_e \) and \( T_e \) but not \( n_1 \). It is important to note that the powers of \( n_e \) which appear explicitly in equations (4.22) are purely arbitrary, since \( \gamma, \alpha, \) and \( S \) are all functions of
In particular, the form of equation (4.22) does not imply that two-body recombination is dominant.

Bates, Kingston, and McWhirter have solved numerically equations (4.6) and (4.21) to obtain the coefficients $\alpha$ and $S$ in equations (4.22) for the radiation-escape situations indicated previously and for a wide range of $n_e$ and $T_e$. These calculations are based on the use of the semiclassical cross sections of Gryzinski (1959) (see Chapter II, Sec. 4) to obtain the rate integrals $\lambda S^2$ and $\lambda S^3$. Although there is some question as to the accuracy of these cross sections, Bates, Kingston, and McWhirter conclude from a comparison with experimental data that the rate integrals calculated therefrom are correct to within a factor of three.

As an example of the results of these calculations, Fig. 4 shows the recombination coefficient $\alpha$ as a function of electron density for a hydrogen plasma at $T_e = 8000^\circ\text{K}$. Two cases are given, one for which all radiation escapes, that is all $\beta$'s are unity, and the other for which the radiation of the resonance Lyman lines is completely absorbed, that is $\beta_{k_1} = 0$, but all other radiation escapes so that all other $\beta$'s are unity. The range of

![Graph showing recombination coefficient $\alpha$ as a function of electron density $n_e$ for different values of $\beta_{k_1}$]
electron density shown is such that both limits of collision-dominated recombination and radiation-dominated recombination are approached for high and low \( n_e \), respectively. In the former case \( \alpha \) is proportional to \( n_e \), whereas in the latter case \( \alpha \) is independent of \( n_e \). Also shown in Fig. 4 is the Hinnov and Hirschberg value of the recombination coefficient from equations (4.22). (Note, however, that in Fig. 4 the Hinnov and Hirschberg formula is applied above the intended limit of 3000\(^\circ\)K.) In Fig. 5 are

\[ n_1, \text{ cm}^{-3} \]

![Figure 5. The steady-state \((n_1^*)\) and equilibrium \((n_1^0)\) ground-level densities for a hydrogen plasma at \( T_e = 8000\,\text{K.} \)]

\[ n_1^0, \text{ BKM} \quad \beta_{\alpha 1} = 1 \]
\[ n_1^0, \beta_{\alpha 1} = 0 \]
\[ n_1^0 - \frac{n_1^0}{n_1^0} \beta_{\alpha 1} \]

given the values \( n_1^0 \) of the ground-level density in steady state for the two cases of Fig. 4 and the equilibrium value \( n_1^* = (n_1/n_2)^x n_2^x \) of the ground-level density. The ionization coefficient \( S \) in equation (4.22) can be readily found in terms of \( \alpha \) and \( n_1^0 \) from the relation

\[ S = \frac{n_e}{n_1^0} \alpha. \quad (4.23) \]

Figs. 4 and 5 show not only the effect of different degrees of radiation escape on the recombination coefficient and the accuracy of the Hinnov and Hirschberg formula in comparison with the more exact calculations of Bates, et al., but also the effect of radiation escape on the steady-state densities. In this context we note that a larger value of \( n_1^0 \) for a given \( n_e \)
is equivalent to a smaller value of \( n_e \) for a given \( n_1 \) at steady state. Thus, since \( n_1^0 > n_1^* \), we have at steady state \( n_1^2/n_1 < (n_1^2/n_1)^* \), as discussed in the previous section.

It is also of interest to examine the populations \( n_k \) of the various levels, which are calculated from equations (4.6) and which through equations (4.21) and (4.22) are required to obtain results such as those of Figs. 4 and 5. Values of \( n_k \) at \( T_e = 8000^\circ \text{K} \) for several electron densities as given by Bates and Kingston (1963) are shown in Fig. 6, for the case of complete radiation escape. To simplify the presentation the values shown apply to the situation where \( n_1^2/n_1 > (n_1^2/n_1)^* \) as would be the case for a recombining plasma. It can be seen from this figure that the assumption that the high-lying levels are in equilibrium with the free electrons is verified.

For the case of complete radiation escape the formulation of Bates, et al., which yields the results of Figs. 4, 5, and 6, is completely in accord with the preceding discussion of this section. When the radiation of the Lyman lines is absorbed, on the other hand, they allow in their calculations for the possibility that \( n_2 \neq 0 \). The results of these calculations show, however, that for most cases \( n_2 \to 0 \) in a time very much less than that required for
$n_e \to 0$. Under these conditions their results, in particular those shown in Fig. 4, reduce to those that would have been obtained if $n_e$ had been set equal to zero initially.

We conclude this section with a comparison, shown in Fig. 7, of the recombination coefficients calculated by Bates, et al. and by Hinnov and Hirschberg with experimental data for cesium and potassium plasmas measured by Aleskovskii (1963) and by Cool and Zukoski (1966). (The values shown here for Bates, et al. are based on $n_e = 10^{13}$ cm$^{-3}$, which corresponds approximately to $n_e$ in the experiments. The calculated values of $\alpha/n_e$ do not vary greatly with $n_e$, particularly for $T_e \leq 2000$°K, nor do

![Graph showing comparison of recombination coefficients](image)
they differ significantly from the values for a “pseudoalkali” plasma given in Table 4A of Part I of their paper.) It must be recognized that although the theoretical calculations are based on hydrogenic atoms, the experimental data are for alkali-metal plasmas. In view of the differences in structure between the alkali-metal atoms and hydrogen and the differences between the two sets of data, the comparison can be regarded as reasonable.

The discussion and results of this section have been based on the assumption that the free-electron distribution function can be taken as Maxwellian. In the following section we consider the more general case in which the Boltzmann equation and the rate equations are investigated simultaneously.

Exercise 4.1. Find the expression for \( \dot{n}_e \) for use in the electron continuity equation (4.4) that applies to the two-level model of the previous section. Express your result solely in terms of \( n_e, n_1 \), the rate integrals, such as \( \int S^2 \), which depend on \( T_e \), and the radiation-escape parameters. Show that when radiative processes can be neglected, your result can be written in the form of equation (4.13), so that \( n_e^2/n_1 = (n_e^2/n_1)^* \) when \( \dot{n}_e = 0 \).

Exercise 4.2. With the use of numerical approximations consistent with the condition \( N_q \gg 1 \), derive equation (4.18) of the Hinnov and Hirschberg analysis. Show that these approximations are justified for \( T_e \lesssim 3000^\circ \text{K} \).

Exercise 4.3. Show that regardless of equations (4.16), equation (4.15) is not obtained by setting \( p = q + 1 \) in equation (4.9), unless \( \lambda' \) is replaced by \( \lambda \). Explain this apparent contradiction with the use of Exercise 3.5 and the two-level model.

Exercise 4.4. Derive equation (4.21) for \( \dot{n}_1 \), (a) directly from physical considerations, and (b) from equations (4.5) and (4.6) for \( \dot{n}_e \) and \( \dot{n}_{k>1} \).

5. THE EFFECTS OF A NON-MAXWELLIAN DISTRIBUTION FUNCTION

In our discussion of ionizational nonequilibrium, we have assumed up to this point that as a result of electron-electron collisions the free electron distribution function differs only slightly from a local, isotropic Maxwellian function. Accordingly, our formulation of the rate equations for \( \dot{n}_e \) and \( \dot{n}_k \) has been uncoupled from the Boltzmann equation for \( f^0 \). The electron distribution function in a collision-dominated plasma will be Maxwellian if the conditions of Chapter VIII, Sec. 3, hold for the electrons that participate in exciting and ionizing collisions as well as for thermal electrons. Here we