2. CONTINUITY AND RATE EQUATIONS

In a uniform plasma the continuity equation for the population, that is, number density, of atoms in a given bound electronic energy level expresses a balance between the rates of collisional and radiative excitation and de-excitation of the level and the time rate of change of the number density of the level. Similarly, the continuity equation for the free electron density involves the rates of ionization and recombination. In a nonuniform plasma the effects of convection and diffusion must be added to these equations. If, as in Chapter VIII, Sec. 8, \( n_k \) denotes the number density of atoms in the \( k \)th electronic level and \( \dot{n}_k \) the corresponding net volumetric rate of production of such atoms as a result of nonelastic collisional and radiative processes, the continuity equation for the \( k \)th level is

\[
\frac{\partial n_k}{\partial t} + \nabla \cdot [n_k (u + U_k)] = \dot{n}_k. \tag{2.1}
\]

As before, \( u \) is the mass velocity of the plasma as a whole and \( U_k \) is the diffusion velocity of atoms in the \( k \)th level. The corresponding continuity equation for the electron number density is

\[
\frac{\partial n_e}{\partial t} + \nabla \cdot [n_e (u + U_e)] = \dot{n}_e. \tag{2.2}
\]

A similar equation applies to the ions. These equations can be viewed as derived directly from the physical requirements of continuity for \( n_k \) and \( n_e \) or, alternatively, as obtained from the integration over velocity space of the excited-atom and electron Boltzmann equations. In the latter interpretation the terms \( \dot{n}_k \) and \( \dot{n}_e \) represent the velocity integrals of nonelastic-collision terms.

In this section we consider the formulation of the expressions for \( \dot{n}_k \) and \( \dot{n}_e \). The equations for these quantities are called rate equations. For clarity and simplicity we restrict our attention to a quasineutral plasma with a single ionizable species, so that the \( k \)th level refers to an electronic level of that species. Furthermore, we neglect multiple ionization and assume that a negligible fraction of the ions are excited above the ionic ground level. Although we use the word level to denote a degenerate set of indistinguishable quantum states of identical energy, in practice the continuity and rate equations are often applied to groups of such levels of nearly the same energy. That is, the \( k \)th "level" would then represent several actual levels taken altogether. The utility of this grouping of levels results from the efficiency of collisional equilibration among neighboring levels.

Before discussing the general rate equations, to fix ideas we first consider a useful simplified version of the rate equation for the first excited level.
In many cases, the population \( n_2 \) of this level is determined primarily by the rates of collisional and radiative transitions to and from only the ground level \( (k = 1) \). As indicated previously, we restrict our attention to situations where collisional excitation and de-excitation result from electron-atom collisions. The rate per unit volume of inelastic excitation collisions of the type \( e + A_1 \rightarrow e + A_2 \), where \( A_1 \) and \( A_2 \) represent atoms in the first and second levels, will be proportional to the product \( n_e n_1 \), and can be written as

\[
n_e n_1 S^2.
\]

Under conditions for which the departure of the electron distribution function from an isotropic distribution \( f^0 (C) \) is small, the rate integral \( S^2 \) is given in terms of the excitation cross section \( Q_1 \rightarrow 2 \) by

\[
S^2 = \int_{\epsilon_1/2}^{\epsilon_2} f^0 (C) Q_1 \rightarrow 2 (C) 4\pi C^2 dC.
\]

Here \( \epsilon_1 \) is the excitation energy measured from the ground level and \( Q_1 \rightarrow 2 \) is the total cross section, integrated over all deflection angles. We have accounted for the consequences of the small electron mass by writing the relative speed \( g \) as \( C \) and \( Q_1 \rightarrow 2 (g) \) as \( Q_1 \rightarrow 2 (C) \) and have integrated over all velocities of the atoms. The lower limit on the integration reflects the fact that for excitation the electron kinetic energy \( \frac{1}{2} m_e C^2 \) must be greater than \( \epsilon_1 \). The cross section \( Q_1 \rightarrow 2 \) and the other nonelastic cross sections that enter the present calculations are discussed in Chapter II, where, in Sec. 14, representative values are given.

It is conventional to rewrite the expression for \( S^2 \) in terms of the electron energy \( \epsilon = \frac{1}{2} m_e C^2 \). If we assume that the electron distribution is Maxwellian, that is, that

\[
f^0 = f_M (\epsilon) = \left( \frac{m_e}{2\pi kT} \right)^{3/2} e^{-\epsilon/kT},
\]

we then have

\[
S^2 = \int_{\epsilon_1}^{\infty} f_M (\epsilon) Q_1 \rightarrow 2 (\epsilon) \frac{8\pi}{m_e^2} \epsilon d\epsilon. \tag{2.3}
\]

In the inverse process, collisional de-excitation of the first excited level results from superelastic collisions of the type \( e + A_2 \rightarrow e + A_1 \). The rate per unit volume of such collisions is

\[
n_e n_2 S_1,
\]

where

\[
S_1 = \int_{0}^{\infty} f_M (\epsilon) Q_2 \rightarrow 1 (\epsilon) \frac{8\pi}{m_e^2} \epsilon d\epsilon. \tag{2.4}
\]
In collisions of this type, since $\epsilon_2 > \epsilon_1$, the lower limit on the integration is zero. The energy difference $\epsilon_{12}$ is imparted to the free electron, hence the designation superelastic.

In addition to the foregoing, radiative processes are important in the rate equation for $n_2$. Thus, for example, in a uniform steady-state plasma whether or not $n_2/n_1$ will equal its equilibrium Boltzmann value will depend on the relative magnitudes of radiative and collisional processes. The volumetric rate of depopulation of $n_2$ as a result of the spontaneous emission of radiation in the transition $A_2 \rightarrow A_1 + h\nu$ is

$$n_2 A_{21}.$$ 

Here $A_{21}$, an atomic constant with dimensions of sec$^{-1}$, is the Einstein $A$ coefficient for the transition in question (see Chapter II, Sec. 9). (Some authors define $A_{21}$ in terms of the radiative emission per unit solid angle and hence would write $4\pi n_2 A_{21}$ in place of $n_2 A_{21}$ for the depopulation rate.) The net rate of radiative transitions between the levels in question depends on the rates of absorption and induced emission as well as on the rate of spontaneous emission. We shall discuss the role of these additional processes later. For present purposes it is sufficient to write the local net rate of radiative transitions as

$$n_2 A_{21} \beta_{21}.$$ 

The radiative-escape factor $\beta_{21}$, which is defined more precisely by equation (2.19) to follow, will be zero if locally the rate of upward $1 \rightarrow 2$ radiative transitions just counterbalances the rate of downward $2 \rightarrow 1$ radiative transitions and will be unity if the rates of absorption and induced emission can be neglected in comparison with the rate of spontaneous emission.

Combining the foregoing expressions, we obtain the simplified rate equation for $n_2$ in the form

$$n_2 = n_e n_1 S^2 - n_2 (n_e S_1 A_{21} \beta_{21}).$$  \hspace{1cm} (2.5)$$

This equation can be simplified somewhat by considerations of detailed balancing (see Chapter II, Sec. 11). In general, detailed balancing applies only to the relationship between cross sections, such as $Q^{1 \rightarrow 2}$ and $Q^{2 \rightarrow 1}$, which are atomic constants. If, however, the distribution function is Maxwellian, the quantities $S^2$ and $S_1$ are also related by the requirements of detailed balancing. At equilibrium, the rate $n_e n_1 S^2$ of collisional excitation of the first excited level in upward, inelastic $1 \rightarrow 2$ transitions is just balanced by the rate $n_e n_2 S_1$ of collisional de-excitation in downward,
superelastic $2 \rightarrow 1$ transitions. Accordingly, we have [see Exercise (II 11.6)]

$$\frac{1}{2} S_1^{2} = \left( \frac{n_2}{n_1} \right)^* = \frac{g_2}{g_1} e^{-n_2/kT_e}, \quad (2.6)$$

where $(n_2/n_1)^*$ is the ratio of densities at equilibrium and $g_2$ and $g_1$ are the degeneracies of the levels. The use of $T_e$ on the right-hand side of equation (2.6) is based on the following consideration: For excitation and de-excitation as a result of collisions with electrons having a Maxwellian distribution function, $1/S_1^{2/2}$ will be a function only of atomic constants and the electron temperature $T_e$. This function can be evaluated at equilibrium, where $T_e = T$ and where for a balance between corresponding upward and downward transitions equation (2.6) must hold.

With the use of equation (2.6) the rate equation (2.5) becomes

$$\dot{n}_2 = n_e n_1 S^2 - n_2 \left[ \frac{n_e S^2}{(n_2/n_1)^*} + A_{21} \beta_{21} \right] \quad (2.7a)$$

or

$$\dot{n}_2 = n_e n_1 S^2 \left[ 1 - \frac{n_2/n_1}{(n_2/n_1)^*} \right] - n_2 A_{21} \beta_{21}. \quad (2.7b)$$

Under conditions of complete equilibrium, where $n_2/n_1 = (n_2/n_1)^*$ and by detailed balancing $\beta_{21} = 0$, $\dot{n}_2$ is zero. For nonequilibrium situations it can be seen from equations (2.5) and (2.7) that the relative importance of radiative and collisional rates of depopulation of the first excited level is given by the ratio

$$\frac{A_{21} \beta_{21}}{n_e S_1^{2}} = \frac{A_{21} \beta_{21} (n_2/n_1)^*}{n_e S_1^{2}}. \quad (2.8)$$

If this ratio is small, radiative processes can be neglected.

The extension of the previous considerations to the general rate equations for the $\dot{n}_k$ and $\dot{n}_e$ is a matter of straightforward bookkeeping. In the general situation, however, we must also account for the rates of ionization and recombination from and to the energy level in question. The volumetric rates of collisional ionization and three-body recombination of the form $e + A_k \rightarrow e + A^+$ are given by

$$n_e n_k S^2$$

and

$$n_e^2 n_1 \lambda S_k = n_e S^2 \lambda S_k,$$
where the symbol $\lambda$ denotes the so-called continuum of states associated with the free electrons. In terms of the ionization cross section $Q^{k-\lambda}$, we have

$$\lambda S^{\lambda} = \int_{\epsilon_{k\lambda}}^{\infty} f_M(\epsilon) Q^{k-\lambda} \frac{8\pi}{m_e^2} \epsilon \, d\epsilon. \quad (2.9)$$

Here $\epsilon_{k\lambda}$ is the ionization potential from the $k$th level. The corresponding expression for $\lambda S_k$ in terms of the cross section $Q^{i-k}$ for three-body recombination is [see Exercise (II 11.4)]

$$\lambda S_k = \int_{0}^{\infty} \int_{0}^{\infty} f_M(\epsilon) f_M(\epsilon') q^{i-k}(\epsilon, \epsilon') \frac{32\sqrt{2\pi^2}}{m_e^{7/2}} \epsilon \sqrt{\epsilon'} \, d\epsilon \, d\epsilon'. \quad (2.10)$$

where $\epsilon'$ denotes the energy of the second electron in the collision. A peculiarity of this process is that the cross section $q^{i-k}$, as used here, has dimensions of cm$^5$ rather than of cm$^3$. For a Maxwellian distribution, however, the use of equation (2.10) is not necessary, since by detailed balancing we have directly the relation [see Exercise (II 11.6)]

$$\lambda S_k = \left( \frac{n_e^2}{n_k} \right)^* = \left( \frac{2\pi m_e kT_e}{\hbar^2} \right)^{3/2} \frac{2g_i}{g_k} e^{-\epsilon_{k\lambda}/kT_e}. \quad (2.11)$$

Here $g_i$ is the degeneracy of the ground level of the ion and the Saha equation (II 10.5a) for the $k$th atomic level has been used to write $(n_e^2/n_k)^*$ in terms of $T_e$.

We can now write the collisional contribution to $\dot{n}_k$ as

$$\dot{n}_{k,\text{coll}} = n_e \left[ \sum_{i<k} n_i \lambda S^k + \sum_{i>k} n_i \lambda S_k + n_e \lambda S_k \right] + n_k \left( \sum_{i<k} \lambda S_i + \sum_{i>k} \lambda S_i + \lambda S^k \right) \quad (2.12a)$$

or, alternatively, as

$$\dot{n}_{k,\text{coll}} = n_e \left[ \sum_{i<k} n_i \lambda S^k \left( 1 - \frac{n_k/n_i}{(n_i/n_k)^*} \right) + n_k \sum_{i>k} \lambda S_i \left( \frac{n_i/n_k}{(n_i/n_k)^*} - 1 \right) \right] + n_k \lambda S^k \left( \frac{n_e^2/n_k}{(n_e^2/n_k)^*} - 1 \right). \quad (2.12b)$$

In equation (2.12a) the first three terms in the brackets account for the collisional excitation to the $k$th level as a result of inelastic transitions from lower levels, superelastic transitions from higher levels, and three-body recombination, respectively. The last three terms account for the collisional de-excitation from the $k$th level as a result of the inverse processes—superelastic transitions to lower levels, inelastic transitions to higher levels, and collisional ionization, all from the $k$th level. Equation
(2.12b) incorporates the detailed balancing relationships between the rate integrals and exhibits the fact that \( n_{k\text{coll}} = 0 \) when the ratios of the number densities have equilibrium values.

The rate integrals \( j_{sk} \) and \( k_{sj} \) in equations (2.12) are given in terms of the relevant cross sections by the following equations, which correspond to equations (2.3) and (2.4) for \( 1 \rightarrow 2 \) transitions:

\[
 j_{sk} = \int_{\epsilon_{jk}}^{\infty} f_{M}(\epsilon) Q_{1 \rightarrow 2}(\epsilon) \frac{8\pi}{m_e^2} \epsilon \, d\epsilon \tag{2.13}
\]

and

\[
 k_{sj} = \int_{0}^{\infty} f_{M}(\epsilon) Q_{2 \rightarrow 1}(\epsilon) \frac{8\pi}{m_e^2} \epsilon \, d\epsilon. \tag{2.14}
\]

The detailed-balance relation between \( j_{sk} \) and \( k_{sj} \), which as before, holds only for \( f^0 = f_{M} \), is (cf. equation (2.6))

\[
 \frac{j_{sk}}{k_{sj}} = \left( \frac{\epsilon_{kj}}{\epsilon_{jk}} \right) e^{-(\epsilon_{kj}/kT_e)} \tag{2.15}
\]

Equations (2.13) through (2.15) also apply to the rate integrals \( j_{sk} \) and \( k_{sj} \) with the appropriate change of subscripts and superscripts.

To complete the rate equation for \( n_k \) we must account, as in equation (2.5), for radiative processes. The rate per unit volume of depopulation of the \( k \)th level as a result of the spontaneous emission of radiation in the transition \( A_k \rightarrow A_j + \hbar \nu \) is

\[
 n_k A_{kj}, \tag{2.16}
\]

where \( A_{kj} \) is the Einstein \( A \) coefficient for this transition. The corresponding volumetric rate of population of the \( k \)th level as a result of the absorption of radiation in the inverse transition \( A_j + \hbar \nu \rightarrow A_k \) is given by

\[
 n_j B_{jk} \int_{\nu} \frac{\phi_{jk}(\nu)}{\int_{0}^{\infty} I_{\nu} \, d\Omega} \, d\nu. \tag{2.17}
\]

Here \( B_{jk} \) is the Einstein \( B \) coefficient, discussed in Chapter II, Secs. 9 and 11, which is related to \( A_{kj} \) by \( A_{kj} = 4\pi B_{jk} = 2g_j/g_k \hbar^2 c^2 \) (\( h \) is Planck's constant and \( c \) is the speed of light). The quantity \( \int_{0}^{\infty} I_{\nu} \, d\Omega \) is the specific intensity of radiation, integrated over all solid angles, and is proportional to the local radiant energy density at the frequency \( \nu \). The line-shape factor \( \phi_{jk}(\nu) \) accounts for the fact that as a result of line broadening the frequency of the radiation associated with the transition in question is not precisely fixed but varies over a narrow range. The factor \( \phi_{jk}(\nu) \) has the property that \( \int \phi_{jk}(\nu) \, d\nu = 1 \), where the integration is extended over all frequencies of the particular line. [Strictly speaking, this normalization of \( \phi_{jk} \)
applies to the extent that the equilibrium value of \( I_v \), as given by the Planck function \( B_\nu \), is approximately independent of \( \nu \) over the effective range of the line. More precisely, to insure detailed balancing at equilibrium, we require that \( \int \phi_{jk}(\nu)B_\nu(\nu)\,d\nu = B_\nu(h\nu = \varepsilon_{jk}) \), where \( \varepsilon_{jk} \) is fixed. See Exercise 2.4.

In addition to transitions associated with absorption and spontaneous emission, we must in general account for the phenomenon of induced emission in which downward, \( k \rightarrow j \) transitions are induced, or stimulated, by the presence of the radiation field. The rate of such transitions is proportional to \( n_k \) and to the specific intensity \( I_v \). If we assume that the line-shape factor for induced emission is the same as that for absorption, this rate is given by the expression (2.17) with \( n_j \) replaced by \( n_k g_j/g_k \). The combined effects of absorption and induced emission can then be written as the net volumetric transition rate

\[
\left(n_j - \frac{g_j}{g_k}n_k\right)B_{jk}\int \phi_{jk}(\nu)\int_0^{4\pi} I_v \, d\Omega \, d\nu.
\]

(2.18)

We note in passing that if there exists a population inversion in a nonequilibrium situation such that \( n_k g_j/g_k > n_j \), the specific intensity is augmented by the excess of induced emission over absorption and the radiation field gains energy at the expense of the plasma. Under suitable conditions this effect can lead to laser action. (In this case a more appropriate form of the expression (2.18) is obtained by removing the two integrations.)

Since the specific intensity \( I_v \) depends on the emission and absorption of radiation throughout the plasma, the evaluation of expression (2.18) is coupled with considerations of radiative transfer. To avoid (or rather to make less explicit) this complexity, we introduce the local radiation-escape parameter \( \beta_{kj} \) defined by

\[
(1 - \beta_{kj})n_k A_{kj} \equiv \left(n_j - \frac{g_j}{g_k}n_k\right)B_{jk}\int \phi_{jk}(\nu)\int_0^{4\pi} I_v \, d\Omega \, d\nu.
\]

(2.19)

In terms of \( \beta_{kj} \), the net contribution to \( n_k \) as a result of the emission and absorption of radiation in \( j \leftarrow k \) transitions is, from expressions (2.16), (2.18), and (2.19),

\[
-n_k A_{kj} \beta_{kj}.
\]

(2.20)

Under conditions for which \( \beta_{kj} \approx 0 \), as when the plasma is optically thick, there exists a local balance between emission and absorption. In particular, such a balance exists for local quasiequilibrium values of \( n_k/n_j \) and \( I_v \). In many laboratory plasmas radiation which corresponds to
transitions for which \( j > 1 \) escapes from the plasma, and in this case \( \beta_{kj} \approx 1 \). Transitions for which \( j = 1 \) terminate in the ground level; the corresponding radiation is described as resonance radiation. Since the density of absorbing ground-level atoms is relatively large, the escape parameter \( \beta_{kl} \) for resonance radiation is often small. Nevertheless, because \( A_{k1} \) is typically large, the product \( A_{k1} \beta_{k1} \) may not be negligible in comparison with the corresponding collisional rate \( n_e A_{k1} \), particularly at lower values of \( n_e \). The parameters \( \beta_{kl} \), as well as the other values of \( \beta_{kj} \) if they are not known to be approximately unity, must be calculated for a particular application from the theory of radiative transfer.

We now consider the so-called free-bound radiative transitions that are associated with the photoionization and two-body recombination reactions \( hv + A_k \rightarrow e + A^+ \). The rate \( n_e n_j A_{jk} \) per unit volume of two-body recombination reactions is given in terms of the corresponding cross section \( Q^{k \rightarrow l}(\varepsilon) \) by (see Chapter II, Sec. 11, and Exercise II 11.5)

\[
n_e n_j A_{jk} = n_e n_j \int_0^\infty f_{md}(\varepsilon) Q^{k \rightarrow l}(\varepsilon) \frac{8\pi}{m_e} \varepsilon \, d\varepsilon. \tag{2.21}
\]

We account for the absorption of radiation in the photoionization transition \( hv + A_k \rightarrow e + A^+ \) by the use of the radiation escape parameter \( \beta_{jk} \), where \( 1 - \beta_{jk} \) is defined as the local ratio of the rate of photoionization reactions to the rate of two-body recombination reactions. The net contribution of these processes to \( n_k \) is then

\[
+ n_e^2 A_{jk} \beta_{jk}, \tag{2.22}
\]

where we have replaced \( n_e n_j \) by \( n_e^2 \).

Combining expressions (2.20) and (2.22) and summing over all relevant transitions, we have for the total radiative contribution to \( \dot{n}_k \)

\[
\dot{n}_{rad} = -n_k \sum_{j<k} A_{kj} \beta_{kj} + \sum_{l>k} n_l A_{lk} \beta_{lk} + n_e^2 A_{jk} \beta_{jk}. \tag{2.23}
\]

Here the second and third terms on the right account for downward transitions that terminate in the \( k \)th state.

We can now combine equations (2.12a) and (2.23) to form an expression for the net rate of change of \( n_k \) as a result of collisional and radiative processes. Accordingly, we have

\[
\dot{n}_k = n_e \sum_{j<k} n_j S^k - n_k \left[ \sum_{j<k} (n_e S_{kj} + A_{kj} \beta_{kj}) + n_e \left( \sum_{l>k} S^l + S^k \right) \right] + n_l (n_e S_k + A_{lk} \beta_{lk}) + n_e^2 (n_e S_k + A_{jk} \beta_{jk}). \tag{2.24}
\]

Here we have chosen to group terms with respect to their dependence on \( n_j, n_k, n_l, \) and \( n_e^2 \). We note that the coefficients \( S_{kj}, S^k, \) and \( S_k \) can be
Section 2 Continuity and Rate Equations

expressed in terms of $j'S^k$, $S^l$, and $S^4$ by the use of equations (2.11) and (2.15).

A similar analysis leads to the corresponding expression for $n_e$. In fact, we have already considered all the processes that contribute to $n_e$, since these processes involve transitions to and from the bound levels. All that remains in the evaluation of $n_e$ is a matter of bookkeeping. The result is

$$n_e = \sum_k \left[ n_e n_{k,4} S^k - n_e^2 (n_e^2 S_k + A_{jk} \beta_{jk}) \right].$$

(2.25)

In accord with the observation that the processes that contribute to $n_e$ have been considered in the evaluation of $n_k$, it can be verified from equations (2.24) and (2.25) that

$$\dot{n}_e + \sum_k \dot{n}_k = 0.$$ 

(2.26)

This reflects the physical fact that for the situation that we have considered where $n_e = n$, the total number of heavy particles, atoms plus ions, of the species in question is not changed by the nonelastic collisional and radiative processes. As a result of equation (2.26), one of the set of rate equations for $n_k$ and $\dot{n}_e$ must be regarded as redundant.

Having obtained expressions for $n_k$ and $\dot{n}_e$, we next turn our attention to the application of these expressions to the study of the effects of radiation escape on departures of the populations of the various excited levels and of the degree of ionization from their equilibrium values.

Exercise 2.1. Find from equations (2.7) an expression for $n_2/n_1$ in a uniform steady-state plasma for which, by equation (2.1), $n_2 = 0$.

Exercise 2.2. Consider a transient situation in the absence of spatial gradients for which $n_e$, $n_1$, and $T_e$ are constant. Initially, at time $t = 0$, $n_2$ is elevated above its equilibrium value. With the use of the approximate form (2.7) for $\dot{n}_2$, find an expression for $n_2$ as a function of time. What is a characteristic relaxation time for this process? How would your results differ if $n_2$ were initially depressed below its equilibrium value?

Exercise 2.3. Write the "chemical" equations, such as $e + A_j \rightarrow e + A_k$, for each type of transition that enters the rate equations (2.12a), (2.24), and (2.25).

Exercise 2.4. Verify that $\beta_{kj} = 0$ at equilibrium, when $n_k/n_j = (n_k/n_j)^*$ and the specific intensity of radiation $I_\nu$ is given by the Planck function

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\nu/kT} - 1}.$$
Exercise 2.5. Find expressions for a set of ratios, similar to those given by equation (2.8), such that if these ratios are each small, radiative processes can be neglected in the rate equations (2.24) and (2.25).

3. DEPARTURES FROM THE SAHA EQUATION

In this section we consider the equations for the level populations \( n_k \) and electron density \( n_e \) for a uniform, steady-state plasma. At first it might seem that under such conditions \( n_k \) and \( n_e \) would always be given by their equilibrium values. As a result of radiation escape, however, there may exist a nonequilibrium situation such that the ratios \( n_k/n_1 \) are not equal to \((n_k/n_1)^*\) as given by equation (2.15) and the ratio \( n_e^2/n_1 \) is not equal to the value \((n_e^2/n_1)^*\) given by the Saha equation (2.11).

Before considering the general rate equations as applied to uniform, steady-state plasmas, we shall discuss the consequences of a simple approximate formulation, the two-level model. In this model, the atoms of the ionizable species are represented in terms of only two levels, the ground level and a single excited level. From equations (2.24) and (2.25) the pertinent rate equations, those for \( \dot{n}_2 \) and \( \dot{n}_e \), are then

\[
\dot{n}_2 = 0 = n_e n_1 S^2 - n_2 (n_e^2 S_1 + A_{21} \beta_{21} + n_e^2 S^3) + n_e^2 (n_e^2 S_2 + A_{22} \beta_{22})
\]

and

\[
\dot{n}_e = 0 = n_e (n_1 S^4 + n_2 S^4) - n_e^2 (n_e^2 S_1 + A_{11} \beta_{11} + n_e^2 S_2 + A_{12} \beta_{12}).
\]

Here \( \dot{n}_2 \) and \( \dot{n}_e \) are set equal to zero since the left-hand sides of the continuity equations for \( n_2 \) and \( n_e \) are zero for a uniform, steady-state plasma. Although the two-level model should not be regarded as an accurate representation, it can be used to demonstrate the principal qualitative effects that we wish to discuss in this section. The excited level in this model can be taken as either the actual first excited level of the ionizable species in question or as a "lumped" level of higher degeneracy selected so as to account approximately for the effects of several lower-lying levels.

It is convenient as a further approximation to neglect in the rate equation (3.1) for \( \dot{n}_2 \) all transitions between the excited level and the continuum. This simplification, which leads to the rate equations (2.5) and (2.7) of the previous section, will be valid if the rates of population and, separately, depopulation of \( n_2 \) in transitions from and to the continuum are small in comparison with the rates of transitions from and to the ground