and accelerators. On the other hand, if the effects of thermal diffusion are neglected, the rigorous expression (4.11) for the current density is

$$J_e = \sigma_\parallel \mathbf{E}_\parallel + \sigma_\perp \mathbf{E}_\perp + \sigma_H \mathbf{b} \times \mathbf{E}.$$  (7.9)

For comparison with equation (7.8), equation (7.9) can be rearranged to yield (see Exercise 7.2)

$$J_e = \sigma_\parallel \mathbf{E}_\parallel + \left(\frac{\sigma_\parallel^2}{\sigma_\perp} + \frac{\sigma_H^2}{\sigma_\perp}\right) \mathbf{E}_\perp - \frac{\sigma_H}{\sigma_\perp} J_e \times \mathbf{b}.$$  (7.10)

Thus the angle between $(J_e)_\parallel$ and $\mathbf{E}_\perp$ is not $\tan^{-1} \beta_e$ but rather $\tan^{-1}(\sigma_H/\sigma_\perp)$ and the conductivity in the direction of $(J_e)_\parallel$ is not $\sigma_\parallel$ but rather $(\sigma_\parallel^2 + \sigma_H^2)/\sigma_\perp$, which has been called the “quasicalar” conductivity $\sigma_Q$. A comparison of $\sigma_H/\sigma_\perp$ with $\beta_e = \omega_d/\nu_e H$ and $(\sigma_\parallel^2 + \sigma_H^2)/\sigma_\perp$ with $\sigma_\parallel$ is shown in Fig. 11 for atmospheric-pressure equilibrium argon at $\beta_e = 5$. It can be seen that the mean-free-path representation for the quantities in question can be significantly in error.

Exercise 7.1. Demonstrate that the Frost formulas (7.3), (7.5), and (7.6) for the various transport properties reduce to the exact expressions in the two limits of weakly ionized and fully ionized plasmas.

Exercise 7.2. Show that equation (7.10) follows from equation (7.9). [Suggestion: first evaluate $J_e \times \mathbf{b}$ from equation (7.9).]

Exercise 7.3. Plot as a function of $\beta_e$ the values of $(\sigma_H/\sigma_\perp)/\beta_e$ and $\sigma_Q/\sigma_\parallel$ for a fully ionized plasma, with the use of the results given by Fig. 7 of Sec. 4.

8. NONELASTIC COLLISIONS

In the foregoing analysis for the electron transport properties, nonelastic collisions have not been considered. Such collisions, however, play an important role in the dynamics of partially ionized gases. For example, as we shall discuss in the following chapter, nonelastic collisions are of primary importance in the electron continuity equation. It is pertinent, therefore, to ask how nonelastic collisions enter the equations of the present analysis and, in particular, to find the additional restrictions which thereby are imposed on the electron transport-property calculations of Sec. 4.

The Cartesian-tensor equations (VII 6.39) and VII (6.40) for $f^0$ and $f^1$ are, when multiplied by the factors $(1/2)m_e C^2 4\pi C^2 d\mathbf{C}$ and $m_e C(4/3)\pi C^2 d\mathbf{C}$, respectively, energy and momentum equations for electrons of the speed class $d\mathbf{C}$. Accordingly, the nonelastic contributions to these equations can be evaluated from a physical basis by adding respectively to the right-hand
Section 8

Nonelastic Collisions

Sides terms which account for the net rate per unit volume at which electrons of the speed class $dC$ gain energy and momentum as a result of nonelastic collisions, divided by the aforementioned factors. Such a derivation of these terms is in accord with that obtained by accounting initially for nonelastic collisions in the electron Boltzmann equation (Viegas and Kruger, 1968).

On the basis of the foregoing, the nonelastic loss term in equation (VII 6.39) for $f^0$ is given by $-n_e f^0 v_{NE}$, where $v_{NE}$ is the nonelastic total collision frequency for electrons of the speed (or equivalently energy) in question, summed over all relevant nonelastic processes. Thus $v_{NE}$ will be a function of electron energy and, for example, the populations of excited atomic states. The nonelastic gain term is more complicated in that it involves integrals of $f^0$. From considerations of detailed balancing, however, the gain term will be of the same order of magnitude as the loss term if the plasma is not too far from equilibrium. Hence, it is reasonable to estimate the importance of nonelastic collisions in equation (VII 6.39) on the basis of the loss term alone. Accordingly, it follows that the condition

$$v_{ee} \gg \sum_l m_e \bar{v}_{eh}/m_h \text{ [equation (3.8)]}$$

for $f^0$ to be Maxwellian must be augmented by the condition

$$v_{ee} \gg v_{NE} \quad (8.1)$$

for electrons of thermal energies.

Even if the inequality (8.1) is satisfied and $f^0$ is Maxwellian, nonelastic collisions may still enter the calculation of $f^0$ through the integrated energy equation for the electron temperature. In the derivation of the energy equation from equation (VII 6.39) the term involving $v_{ee}$ integrates identically to zero, and therefore the term involving $v_{NE}$ must be retained even when $v_{ee} \gg v_{NE}$. The latter term, which accounts for the net rate of nonelastic energy gain by the electrons, must be added to the right-hand side of the electron energy equation (3.16).

If the relevant nonelastic collisions are with atoms in various levels of electronic excitation that communicate energetically only with the free electrons and by the emission and absorption of radiation, we can obtain the net rate $\bar{N}$ per unit volume of nonelastic energy loss from the free electrons by considering an energy balance for the excited atoms. In this context we consider an ion as an atom with an excitation energy equal to the ionization potential $\epsilon_j$. For simplicity, we consider only one species of singly charged ions. Denoting the local net rate per unit volume of radiant energy loss from the plasma by $\bar{R}$ and the net rate per unit volume of increase in energy in excited levels by $\bar{U}$, we have

$$\bar{U} = \bar{N} - \bar{R}.$$
The quantity $\dot{U}$ can be expressed in terms of the net volumetric rate $n_k$ of production of excited atoms as

$$\dot{U} = \sum_k \epsilon_k n_k,$$

where $\epsilon_k$ is the energy of the $k$th level, measured from the ground level. We now introduce the continuity equations for the excited atoms, viz.

$$\dot{n}_k = \frac{\partial n_k}{\partial t} + \nabla \cdot [n_k(u + U_k)]. \quad (8.2)$$

Combining these expressions, we have for the nonelastic energy loss

$$\dot{\mathcal{N}} = \dot{R} + \frac{\partial}{\partial t} \sum_k \epsilon_k n_k + \nabla \cdot \sum_k \epsilon_k (u + U_k). \quad (8.3)$$

In most cases of interest the dominant term in the foregoing summations is the contribution from the ions. Under these conditions, equation (8.3) becomes

$$\dot{\mathcal{N}} \approx \dot{R} + \epsilon_i \frac{\partial n_i}{\partial t} + \epsilon_i \nabla \cdot [n_i(u + U_i)]. \quad (8.4)$$

To obtain the form of the electron energy equation that accounts for nonelastic collisions, the term $-\mathcal{N}$ must be added to the right-hand side of equation (3.16). With equation (8.4) for $\dot{\mathcal{N}}$ it is convenient to use the relations

$$\frac{\partial n_e}{\partial t} + \nabla \cdot [n_e(u + U_e)] = \dot{n}_i,$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot [n_i(u + U_i)] = \dot{n}_e,$$

and, by conservation of charge,

$$\dot{n}_e = \dot{n}_i.$$

To write $\dot{\mathcal{N}}$ in terms of $n_e$ and $-J_e/e = n_e U_e$, [These relations are equivalent to more familiar equation $\nabla \cdot j = -\partial \rho^t/\partial t$, where $j = e(n_i - n_e)u + J_i + J_e$ is the total current density and $\rho^t = e(n_i - n_e)$ is the charge density.] The energy equation (3.16) is then replaced by

$$\frac{D}{Dt} \left[ n_e \left( \frac{5}{2} kT_e + \epsilon_i \right) \right] + n_e \left( \frac{5}{2} kT_e + \epsilon_i \right) \nabla \cdot u + \nabla \cdot (q_e - \epsilon_i J_e/e) - J_e \cdot E'$$

$$= n_e \sum_k \frac{3}{2} k(T_n - T_e) m_n \bar{v}_{ne} - \dot{R}. \quad (8.5)$$
We now turn our attention to the possible nonelastic effects on the calculation of electron transport properties, when \( f^0 = f_M \). In equations (VII 6.40) and (4.1) for \( f \), the nonelastic loss term is given by \( -n_e f f_{NE} \). Insofar as the nonelastic cross sections can be regarded as isotropic functions of scattering direction, the nonelastic gain term will be zero. The analysis for the transport properties would then remain the same as before, with the elastic momentum-transfer collision frequency \( v_{eh} \) replaced by the sum \( v_{eh} + v_{NE} \). Whenever the inequality (8.1) is satisfied, however, \( v_{NE} \) will make a negligible contribution to this sum since, as a result of the ion term, \( v_{eh} \) is at least as great as \( v_{en} \). This conclusion applies even if the nonelastic cross sections are not isotropic, if again it is assumed that the nonelastic gain term is of no greater magnitude than the loss term. Thus, if the inequality (8.1) is satisfied, although important nonelastic effects may enter the integrated electron continuity and energy equations, the electron transport-property calculations of Sec. 4 apply without modification.

In the following chapter we shall discuss the effects of nonelastic collisions in greater detail, with an emphasis there on departures from the Saha equation and the rates of ionization and recombination.

**Exercise 8.1.** Obtain the generalization of the electron energy equation (8.5) that accounts in the nonelastic loss term (8.4) for more than one species of singly charged ions, each with a different ionization potential.

**REFERENCES**


