numerical calculations of DeVoto (1967), who evaluated the \( \eta \) and \( \lambda \) for partially ionized atmospheric-pressure argon, first using the complete formalism of Hirschfelder, Curtiss, and Bird and then neglecting entirely the presence of the electrons [including the term (6.6)] in the evaluation of \( \mu \) and the heavy-particle contribution to \( \lambda \). Over a range of temperatures from 5000°K to 20,000°K, he found less than a 1% change in both coefficients, when the same Sonine-polynomial approximation is used for each method.

The foregoing discussion has been limited, nominally, to the case of zero magnetic field. Extension of the Chapman-Enskog theory of a multi-component plasma to the magnetic-field case has been worked out by DeVoto (1969) and by Bisnovatyy-Kogan (1964) [see Kalikhman (1967)]. Although this theory already incorporates certain features associated with the small mass of the electron, Daybelge (1968) shows that the results can be further simplified in exactly the same way as in the case of zero magnetic field. Except for very strong magnetic fields or low densities, however, the ratio of the ion cyclotron frequency \( eB/m_i \) to the ion collision frequency will be small. When this is so, the simplified equations for the heavy-particle transport properties reduce to those of the zero-magnetic-field case. Accordingly, under these conditions our previous discussion of the use of the results of Hirschfelder, Curtiss, and Bird is directly applicable and the magnetic field enters only in the calculations of Sec. 4 for the electron transport effects and, thereby, in the evaluation of the effective forces (6.6).

Having established the sources that can be used for the calculation of the heavy-particle transport properties, we return in the following section to the consideration of electron transport properties. In particular, with the use of results of Sec. 4, we shall examine the accuracy of various approximate formulas for these properties.

7. MIXTURE RULES

On the basis of the theory of Sec. 4 the electron transport properties can be calculated to any accuracy desired by means of the Sonine-polynomial expansions. These calculations incorporate the simplifications associated with the small mass of the electron and hence are much less complicated than the corresponding classical calculations for a gas mixture. None the less, in many practical cases it is desirable or necessary to use for the transport properties approximate formulas that can be evaluated with a minimum of numerical effort. In this section we discuss such formulas, called mixture rules, and assess their accuracy in comparison with the exact theory of Sec. 4. Throughout the section we shall restrict our attention to electron transport properties.
We first consider the approximate calculation of the parallel electrical conductivity \( \sigma_\parallel \), which applies in the absence of or in the direction parallel to a magnetic field. The simplest expression for \( \sigma_\parallel \) is the mean-free-path formula

\[
\sigma_\parallel \approx \frac{n_e e^2}{m_e \bar{v}_{eH}}.
\]  

(7.1)

In practice, various methods are used to evaluate the collision frequency in equation (7.1), as discussed in Chapter II, Sec. 6. These differences in the form of the collision frequency can be interpreted in terms of different methods of averaging the speed-dependent cross section \( \sigma_{\alpha\beta} \). If, as indicated in equation (7.1), the well-defined expression (3.15) is used for the collision frequency, with \( \bar{v}_{eH} = \sum \bar{v}_{e\alpha} \), the mean-free-path formula (7.1) becomes identical to the first Sonine-polynomial approximation \( \sigma_{\parallel}^{(1)} \) in the theory of Sec. 4. Hence, Figs. 4 and 5 of Sec. 4, which show the convergence of the Sonine-polynomial approximations, can be used to assess the accuracy of equation (7.1). The mean-free-path formula (7.1) is in error by a factor of about two in the limit of a fully ionized plasma and, for some electron-neutral cross sections, by even larger factors in the limit of a weakly ionized plasma. These inaccuracies can be avoided by the use of the approximate conductivity \( \sigma_{\parallel}^{(add)} \), given by

\[
\frac{1}{\sigma_{\parallel}^{(add)}} = \frac{1}{\sigma_{en}} + \frac{1}{\sigma_{ei}}.
\]  

(7.2)

Equation (7.2) is based on the suggestion of Lin, Resler, and Kantrowitz (1955) that the resistivities associated with the neutrals and ions be added. If the Lorentzian expression (2.14a), with \( f^0 = f_M \) and applied only to the neutral species, is used for \( \sigma_{en} \) and the Spitzer-Harm expression (5.1a) is used for \( \sigma_{ei} \), the conductivity \( \sigma_{\parallel}^{(add)} \) is necessarily correct in the limits of both weakly ionized and fully ionized plasmas. Although the formula (7.2) is accordingly preferable to equation (7.1), \( \sigma_{\parallel}^{(add)} \) may still be in error by as much as a factor of about two at intermediate degrees of ionization, as is shown in Fig. 10.

There exists an alternate mixture rule which yields accurate values for \( \sigma_\parallel \) over the complete range of ionization. This is the formula proposed by Frost (1961), in which the Lorentzian conductivity expression (2.14a) is used with a modified electron-ion collision frequency such that the resultant formula reduces for singly charged ions to the exact Spitzer-Harm value (5.1a) in the limit of a fully ionized plasma. The Frost conductivity \( \sigma_{\parallel}^{(f)} \) is given by

\[
\sigma_{\parallel}^{(f)} = \frac{4\pi n_e e^2}{3kT_e} \int_0^\infty \frac{C^2 f_M}{v_e^{(f)}} dC.
\]  

(7.3a)
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where

\[ \nu_{\text{el}}^{(F)} = \sum_n \nu_{\text{el}}^{(1)} + 0.476 \frac{8\pi n_i}{C^2} \left( \frac{m_e}{2kT_e} \right)^{1/2} \left( \frac{e^2}{4\pi e_0 m_e} \right)^2 \ln \Lambda. \]  

(7.3b)

Comparison of the ion contribution to the collision frequency (7.3b) with \( \nu_{\text{el}}^{(1)} \) given by equation (VII 6.37) shows that not only is the numerical factor changed, but in addition the \( C^{-3} \) speed-dependence of \( \nu_{\text{el}}^{(1)} \) is replaced by \( C^{-2}(m_e/2kT_e)^{1/2} \). Although this modification of the speed-dependence of \( \nu_{\text{el}}^{(1)} \) is based on certain calculations of Hwa (1957), there is no rigorous justification for the Frost formula (7.3). Nevertheless, as shown by Schweitzer and Mitchner (1966) and for partially ionized argon in Fig. 10, the Frost formula is remarkably accurate. In fact, for the lower degrees of ionization shown in Fig. 10 the Frost conductivity, which reduces to the exact value in the Lorentzian limit, is presumably more accurate than even the twelfth Sonine-polynomial approximation. This results from the poor convergence of the Sonine-polynomial expansion in the limit of a weakly ionized plasma for the "worst-case" of argon, which was shown in Fig. 4 of Sec. 4. For higher degrees of ionization, where \( \sigma_{\parallel}^{(F)} \) and \( \sigma_{\parallel}^{(1)} \) agree, the Sonine-polynomial expansion converges more rapidly (cf. Fig. 5) and the values of \( \sigma_{\parallel}^{(1)} \) can be taken as exact.

Mixture rules for the approximate calculation of the thermal-conductivity and thermal-diffusion coefficients \( \lambda_{\parallel}, \lambda_{\perp}, \) and \( \phi_{\parallel} \) have not been as well developed as in the case of \( \sigma_{\parallel} \). For these properties the use of the lowest possible Sonine-polynomial approximation is more complicated since this now requires \( \xi = 2 \). The accuracy of this approach is illustrated in Figs. 4 and 5 of Sec. 4. Alternatively, mean-free-path considerations suggest that the thermal conductivity \( \lambda_{\parallel} \) should be given to within a numerical factor by

\[ \lambda_{\parallel} \simeq \frac{n_e k^2 T_e}{m_e \bar{v}_{eH}}. \]  

(7.4)

From the normalization used in Figs. 4 and 10, it can be seen that for argon equation (7.4) gives \( \lambda_{\parallel} \) to within a factor of three. As a result of the change in sign of the thermal-diffusion coefficient with increasing degree of ionization (see Fig. 5), there is no meaningful counterpart for \( \phi_{\parallel} \) to the simple formula (7.4). One can say, however, that the magnitude of \( \phi_{\parallel} \) is not much greater than \( n_e k/m_e \bar{v}_{eH} \), as can be seen from Fig. 5.

Kruger, Mitchner and Daybelge (1968) have proposed mixture rules for \( \lambda_{\parallel}^{(F)} \) and \( \phi_{\parallel}^{(F)} \) which are similar to the Frost conductivity (7.3). The formulas for these quantities, denoted by \( \lambda_{\parallel}^{(F)} \) and \( \phi_{\parallel}^{(F)} \), are

\[ \lambda_{\parallel}^{(F)} = \frac{4\pi}{3} n_e k \int_0^\infty \frac{1}{\nu_{\text{el}}^{(F)}} \left( \frac{m_e C^2}{2kT_e} - \frac{5}{2} \right)^2 C^f \sigma_{\text{M}} \, dC, \]  

(7.5a)
Figure 10. Comparison of the mean-free-path and Frost values for \( \sigma \) and \( \lambda \) with the values given by the twelfth approximation for atmospheric-pressure argon.
where

\[ v_\phi^{(P)} = \sum_n v_\phi^{(1)} + 1.012 \frac{8 \pi n_i}{C^2} \left( \frac{m_e}{2kT_e} \right)^{1/2} \left( \frac{e^2}{4 \pi e_0 m_e} \right)^2 \ln \Lambda, \quad (7.5b) \]

and

\[ \phi^{(P)} = \frac{4 \pi e n_e}{3} \int_0^\infty \frac{1}{v_\phi^{(P)}} \left( \frac{m_e C^2}{2kT_e} - \frac{5}{2} \right) C f_M \, dC. \quad (7.6a) \]

where

\[ v_\phi^{(P)} = \sum_n v_\phi^{(1)} + 0.6776 \frac{8 \pi n_i}{C^2} \left( \frac{m_e}{2kT_e} \right)^{1/2} \left( \frac{e^2}{4 \pi e_0 m_e} \right)^2 \ln \Lambda. \quad (7.6b) \]

From these expressions a “Frost” value \( \lambda^{(P)} \) for the usual thermal conductivity can be calculated from the relation [cf. equations (4.15)]

\[ \lambda^{(P)} = \lambda^{(P)} - T_e \left( \phi^{(P)} / \sigma^{(P)} \right)^2. \quad (7.7) \]

As with the electrical conductivity, the “Frost” values of the thermal conductivity and thermal-diffusion coefficients, calculated from equations (7.5) through (7.7), have been found to agree well with the exact values. This is illustrated for the thermal conductivity of argon in Fig. 10. Again for low degrees of ionization the values of \( \lambda^{(P)} \) are presumably more accurate than those given by the twelfth Sonine-polynomial approximation, as a result of the poor convergence of the Sonine-polynomial expansions for weakly ionized argon.

We can conclude from the foregoing that the mixture rules of the Frost type, being both accurate and reasonably easy to use, are to be preferred for the practical calculation of transport properties in the absence of a magnetic field. In fact, once the accuracy of these formulas has been established, it would appear that the considerable additional complexity of the use of high-order Sonine-polynomial approximations is hardly warranted, particularly in view of the inevitable uncertainties in cross sections. Corrections for smaller \( \Lambda \), as proposed by Daybelge (1968) and (1970), can be made by dividing the second terms in equations (7.3b), (7.5b), and (7.6b) by \( \sigma_{\parallel}/\sigma_{\parallel}^{SU} \), \( \lambda_{\parallel}/\lambda_{\parallel}^{SU} \), and \( \phi_{\parallel}/\phi_{\parallel}^{SU} \) from Fig. 8 of Sec. 5.

The situation with regard to mixture rules in the presence of a magnetic field is more complicated and somewhat less satisfactory as a result of the additional parameter, the field strength, and the increased number of transport properties. As before, when only limited accuracy is required either mean-free-path formulas or the lowest Sonine-polynomial approximation can be used. In the mean-free-path formulas the value of the parallel component of the transport property is multiplied by \( (1 + \beta_2)^{-1} \) and by \( \beta_4(1 + \beta_2)^{-1} \) to obtain the perpendicular and Hall components, respectively.
Although in these expressions $\beta_e$ can be evaluated in various ways, for definiteness we use the form $\beta_e = \omega_e / \bar{v}_{eh}$ based on $\bar{v}_{eh} = \sum_e \bar{v}_{eh}$ and the definition (3.15) of $\bar{v}_{eh}$. Figs. 6, 7, and 9 of Secs. 4 and 5 illustrate the fact that such formulas yield values that are accurate to within a factor of two or three. In the case of the perpendicular thermal diffusion coefficient, however, the complicated change in sign with both increasing degree of ionization and increasing $\beta_e$ makes the mean-free-path formula of limited utility. We note that by equations (4.41) the simple mean-free-path values for the perpendicular and Hall conductivities

$$\sigma_\perp \approx \frac{n_e e^2 / m_e \bar{v}_{eh}}{1 + \beta_e^2}$$

and

$$\sigma_H \approx \frac{(n_e e^2 / m_e \bar{v}_{eh}) \beta_e}{1 + \beta_e^2}$$

are given in Fig. 6 by the curves for $\sigma_\perp^{4}$ and $\sigma_H^{4}$.

To obtain more accurate values of the electrical conductivity in the presence of a magnetic field at least two additional methods are available. First, to the extent that the electron-neutral collision frequency can be described by a power-law dependence on electron speed—in our notation $\sum_e v_{en} \propto C^2 m$—the results of Shkarofsky, Johnston, and Bachynski (1966), Sec. 8.5, can be used to obtain directly the fourth Sonine-polynomial approximations to $\sigma_\perp$ and $\sigma_H$. Since in most cases the fourth approximation will itself be sufficiently accurate, the primary uncertainty in this method results from the power law representation of the electron-neutral collision frequency. The second method, proposed by Schweitzer and Mitchner (1967), consists of an extension of the Frost mixture rule to $\sigma_\perp$ and $\sigma_H$. This method uses the Lorentzian expressions (2.14b) and (2.14c) for $\sigma_\perp$ and $\sigma_H$, incorporating effective electron-ion collision frequencies which are such that $\sigma_\perp$ and $\sigma_H$ reduce to known values in the limit of a fully ionized plasma. These effective collision frequencies depend on the magnetic-field strength and also on the order of the Sonine-polynomial approximation used to calculate $\sigma_\perp$ and $\sigma_H$ in the fully ionized limit. The curves for these collision frequencies given by Schweitzer and Mitchner (1967) are based in part on the third Sonine-polynomial approximations. Somewhat more accurate values, which incorporate the eleventh approximations, are given by Kruger, Mitchner, and Daybelge (1968), who also give similar formulas for $\lambda_\perp$, $\lambda_H$, $\phi_\perp$, and $\phi_H$. Comparison of the results obtained by this method with exact values shows good agreement.

With respect to the accuracy of approximate formulas for the electrical conductivity in the presence of a magnetic field, it is instructive also to
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examine certain consequences of the usual mean-free-path representation of Ohm's law, namely,

\[ \mathbf{J}_e \simeq \sigma_\parallel \mathbf{E} - \beta_e \mathbf{J}_e \times \mathbf{b}, \]  

(7.8)

where \( \mathbf{b} = \mathbf{B}/B \). According to equation (7.8), in the plane perpendicular to the magnetic field the angle between \( (\mathbf{J}_e)_\parallel \) and the generalized electric field \( \mathbf{E}_\perp = \mathbf{E}_\perp + \nabla \perp p_e/en_e \) is given by \( \tan^{-1} \beta_e \). In addition, the current density \( (\mathbf{J}_e)_\parallel \) is equal to the product of the "scalar" electrical conductivity—in our notation \( \sigma_\parallel \)—and the component of the field \( \mathbf{E}_\perp \) in the direction of the current. These approximations are often used in the interpretation of the electrical characteristics of crossed-field devices such as MHD generators.

![Diagram](image)

**Figure 11.** Comparison of the exact and mean-free-path expressions for \( \sigma_\parallel/\sigma_\perp \) and the quasiscalar conductivity \( \sigma_0 = (\sigma_\parallel^2 + \sigma_\perp^2)/\sigma_\perp \) for atmospheric-pressure argon at \( \beta_e = \alpha_0\sqrt{v_e} = 5 \). According to the mean-free-path representation (7.8), the ratios given by both curves should be everywhere equal to unity.
and accelerators. On the other hand, if the effects of thermal diffusion are neglected, the rigorous expression (4.11) for the current density is

\[ J_e = \sigma_\parallel \varepsilon_\parallel + \sigma_\perp \varepsilon_\perp + \sigma_H \mathbf{b} \times \varepsilon. \]  

(7.9)

For comparison with equation (7.8), equation (7.9) can be rearranged to yield (see Exercise 7.2)

\[ J_e = \sigma_\parallel \varepsilon_\parallel + \left( \frac{\sigma_\perp^2 + \sigma_H^2}{\sigma_\perp} \right) \varepsilon_\perp - \frac{\sigma_H}{\sigma_\perp} J_e \times \mathbf{b}. \]  

(7.10)

Thus the angle between \((J_e)_\parallel\) and \(\varepsilon_\perp\) is not \(\tan^{-1} \beta_e\) but rather \(\tan^{-1} (\sigma_H/\sigma_\perp)\) and the conductivity in the direction of \((J_e)_\parallel\) is not \(\sigma_\parallel\) but rather \((\sigma_\perp^2 + \sigma_H^2)/\sigma_\perp\), which has been called the “quasiscalar” conductivity \(\sigma_Q\). A comparison of \(\sigma_H/\sigma_\perp\) with \(\beta_e = \omega_e/\nu_e H\) and \((\sigma_\perp^2 + \sigma_H^2)/\sigma_\perp\) with \(\sigma_\parallel\) is shown in Fig. 11 for atmospheric-pressure equilibrium argon at \(\beta_e = 5\). It can be seen that the mean-free-path representation for the quantities in question can be significantly in error.

**Exercise 7.1.** Demonstrate that the Frost formulas (7.3), (7.5), and (7.6) for the various transport properties reduce to the exact expressions in the two limits of weakly ionized and fully ionized plasmas.

**Exercise 7.2.** Show that equation (7.10) follows from equation (7.9). [Suggestion: first evaluate \(J_e \times \mathbf{b}\) from equation (7.9).]

**Exercise 7.3.** Plot as a function of \(\beta_e\) the values of \((\sigma_H/\sigma_\perp)/\beta_e\) and \(\sigma_Q/\sigma_\parallel\) for a fully ionized plasma, with the use of the results given by Fig. 7 of Sec. 4.

### 8. NONELASTIC COLLISIONS

In the foregoing analysis for the electron transport properties, nonelastic collisions have not been considered. Such collisions, however, play an important role in the dynamics of partially ionized gases. For example, as we shall discuss in the following chapter, nonelastic collisions are of primary importance in the electron continuity equation. It is pertinent, therefore, to ask how nonelastic collisions enter the equations of the present analysis and, in particular, to find the additional restrictions which thereby are imposed on the electron transport-property calculations of Sec. 4.

The Cartesian-tensor equations (VIII 6.39) and VII (6.40) for \(f^0\) and \(f^1\) are, when multiplied by the factors \((1/2)m_e C^2 4\pi C^2 \, dC\) and \(m_e C (4/3)\pi C^2 \, dC\), respectively, energy and momentum equations for electrons of the speed class \(dC\). Accordingly, the nonelastic contributions to these equations can be evaluated from a physical basis by adding respectively to the right-hand