Exercise 5.1. Show by the use of suitable mean-free-path arguments that for a fully ionized plasma with $B = 0$ it is reasonable to expect that the electron contribution to the viscosity and the ion contribution to the thermal conductivity can be neglected. Compare your estimates of $\eta_i$ and $\lambda_\parallel$ with the exact expressions.

Exercise 5.2. Obtain approximate expressions for $\sigma_\parallel$ and $\lambda_\parallel$ by the ad hoc application of the Lorentzian analysis of Sec. 2 to a fully ionized plasma. Your results should agree with the values given in Table 1.

Exercise 5.3. For a fully ionized plasma evaluate $\sigma_\parallel^{(1)}$.

6. HEAVY-PARTICLE TRANSPORT PROPERTIES

In the previous sections, we have discussed almost exclusively the calculation of electron transport properties. To obtain a complete description of transport phenomena in a collision-dominated plasma, however, it is necessary to consider also the heavy-particle transport properties. Thus, for example, the viscous stress depends on the heavy-particle viscosity coefficient and the total heat flux in a partially ionized plasma depends on the heavy-particle, as well as the electron, thermal conductivity.

In the absence of a magnetic field, a rigorous means of calculation of the heavy-particle transport properties—based on the classical Chapman-Enskog theory for gas mixtures—is provided by Hirschfelder, Curtiss, and Bird (1954), Chapter 7. There it is shown that the total viscous stress $\tau_{\parallel}$ and heat flux $q$ and the species diffusion velocities $U_s$ are related to gradients and forces by the following expressions:

$$
\tau_{\parallel} = \eta \left[ \frac{\partial u_x}{\partial x_{\parallel}} + \frac{\partial u_y}{\partial y_{\parallel}} - \frac{2}{3} (\nabla \cdot u) \delta_{\parallel} \right],
$$

(6.1)

$$
q = \frac{5}{2} kT \sum_s n_s U_s - \lambda_\parallel \nabla T - nkT \sum_s \frac{D^T_s}{\rho_s} \delta_s,
$$

(6.2)

and

$$
U_s = \frac{n^2}{n_s \rho - \frac{1}{\rho_s} D^T_s \nabla \ln T,}
$$

(6.3)

The effective driving forces $d_s$ for diffusion are defined by

$$
d_s \equiv \nabla \left( \frac{n_s}{n} \right) + \left( \frac{n_s}{n} - \frac{\rho_s}{\rho} \right) \nabla \ln p
$$

(6.4)
Here $\eta$ is the overall viscosity coefficient, $\lambda'$ is an overall thermal-conductivity coefficient, and $D_x$ and $D_f^e$ are species diffusion and thermal-diffusion coefficients, respectively. The other symbols in equations (6.1) through (6.4) are defined in Chapter VII, Secs. 2 and 3. As in Sec. 4, the thermal-conductivity coefficient $\lambda'$ is not the usual coefficient $\lambda$, which relates the heat flux to the temperature gradient when the diffusion velocities are zero. We note that, since the classical Chapman-Enskog theory applies to quasiequilibrium situations where the temperatures of the various species are the same, only a single temperature appears in these equations.

Hirschfelder, Curtiss, and Bird show how the various transport coefficients in equations (6.1) through (6.4) can be evaluated, with the use of Sonine-polynomial expansions, in terms of the solutions of linear algebraic equations. As in Sec. 4, these solutions can be expressed in the form of ratios of determinants of collision integrals. The order of the determinants involved is equal to the product of the number of species and the order of the Sonine-polynomial approximation; hence, the formulation is complicated at best. To directly apply this theory to a plasma, the electrons and ions must be regarded as on an equal footing with the other species in the gas mixture. The theory can be used, however, to calculate separately the various heavy-particle diffusion and thermal-diffusion coefficients and the heavy-particle contributions to the viscosity coefficient and the thermal-conductivity coefficient $\lambda'$. Nevertheless, each of these calculations still involves electron collision integrals.

The direct application of the results of Hirschfelder, Curtiss, and Bird to the calculation of the heavy-particle transport properties of a partially ionized plasma suffers from two defects. First, the theory applies to quasiequilibrium plasmas with a single temperature for all species and, second, the results do not incorporate the simplifications associated with the small mass of the electron. Hence they are more complicated than necessary. Daybelge (1968) has examined this situation from the point of view of Chapter VII, Sec. 5, where the expansion of the collision terms for small electron mass was discussed. He finds that for a collision-dominated plasma the only significant effect of the electrons on the heavy-particle distribution functions results from a collisional transfer of momentum associated with the diffusion of the electrons. The reasons for this are that the electrons do not significantly affect the heavy-particle viscosity coefficients and that the heavy-particle diffusion, thermal-diffusion, and thermal-conductivity coefficients depend on equations analogous to equations (VII 6.40) or (4.1), which can be interpreted as speed-dependent momentum equations for each heavy-particle species. The effect of collisions with electrons can be represented in each of these equations by the rate per heavy particle of momentum transfer from the electrons to the heavy-particle species in question.
The rate of momentum transfer from the electrons to the heavy particles can be satisfactorily calculated from a consideration of only the zero-order terms in \( m_e/m_h \) in the expansions of Chapter VII, Sec. 5. Thus, since in this approximation the heavy particles can be regarded as stationary, the differential rate per heavy particle of momentum transfer, integrated over the azimuthal angle \( \phi \), is

\[
n_e m_e C (1 - \cos \chi) f_e(C) f_{eh}(C, \chi) 2\pi \sin \chi \ d\chi \ d^3C.
\]

Integration of this expression over all deflection angles and electron velocities yields for the required rate of collisional momentum transfer

\[
n_e m_e \int_{-\infty}^{\infty} C C Q_{eh}^{(1)}(C) f_e(C) \ d^3C \quad (6.5)
\]

where \( Q_{eh}^{(1)} \) is the momentum-transfer cross section defined by equation (VII 4.9). The notable feature of this result (6.5) is that to this approximation the rate of momentum transfer is independent of the heavy-particle velocity and therefore has the same effect as an external force. Accordingly, Daybelge finds that for purposes of calculation of the heavy-particle transport properties, the electron collision terms in the heavy-particle Boltzmann equations can be omitted, if in their place is incorporated on the left-hand side of each Boltzmann equation an effective force given by equation (6.5). We note that the expression (6.5), when multiplied by \( n_h \), is just equal to the integrated rate of momentum loss to the electrons that would be found from the multiplication of the term \( v_eh f_i^i \) in equation (4.1) by \( n_e m_e C (4/3) \pi C^2 \ dC \) and integration over \( C \) [see the second of equations (VII 6.19)].

As a result of the foregoing considerations, Daybelge gives the following simplified prescription for the evaluation of heavy-particle transport effects in partially ionized plasmas: To within an error of the order of \( \sqrt{m_e/m_h} \), equations (6.1) through (6.4) can be used to evaluate the total viscous stress, the heavy-particle heat flux—which is to be added to the electron heat flux (4.12) or (4.14)—and the heavy-particle diffusion velocities. The transport properties that appear in these expressions are to be evaluated according to the methods of Hirschfelder, Curtiss, and Bird. Their formulas, however, are to be applied to a hypothetical gas mixture of only neutral particles and ions at a common temperature \( T \) which may be different from the electron temperature \( T_e \). The electrons are to be ignored entirely in the calculation of the heavy-particle transport properties and omitted from the summations in equations (6.2) and (6.3). In equation (6.4) for the driving force \( d_x \), the effective force (6.5) is to be added to the external force \( F_x \), and the electrons are to be included in the evaluation of \( n, p, \) and the final summation. As a consequence of the foregoing, it is possible to evaluate
the heavy-particle transport properties on the basis of low-order Sonine-polynomial approximations—which, as discussed by Hirschfelder, Curtiss, and Bird, are usually sufficiently accurate—and at the same time use the methods of Sec. 4 to obtain when necessary higher-order approximations for the electron transport properties.

The experienced reader may raise the objection to this prescription that for the hypothetical mixture of neutrals and ions the condition \( \sum \rho_n U_n + \rho_i U_i = 0 \) would apparently be imposed by use of the results of Hirschfelder, Curtiss, and Bird, whereas for the actual plasma the condition on the diffusion velocities must be \( \sum \rho_n U_n + \rho_i U_i + \rho_e U_e = 0 \) [see equation (VII 2.9c)]. Daybelge (1968) shows, however, that this point does not actually present any difficulty and that to order \( \sqrt{m_e/m_h} \) no error is thereby introduced in the calculation of the heavy-particle transport properties.

To carry out these simplifications in detail it is necessary to evaluate the effective forces (6.5) on the heavy particles. Since the analysis of Sec. 4 for the electron distribution function is uncoupled from the calculation of the heavy-particle distribution functions, these effective forces can be found independently, from the relation [see equations (VII 6.19)]

\[
n_e m_e (CCQ^{(1)}_{eh}) = n_e m_e \frac{4\pi}{3} \int_0^\infty C^4 Q^{(1)}_{eh} f^1 dC.
\]

With the use of equation (4.4) and the subsequent results of Sec. 4, \( f^1 \) can be expressed in terms of the Sonine-polynomial expansion coefficients, which in turn can be found from the solution of equations (4.25) and (4.26). Evaluation of the integral on the right-hand side of equation (6.6), as described in detail by Daybelge, is thus reasonably straightforward. If, however, the heavy-particle diffusion velocities are not small when compared with the electron diffusion velocity this procedure must be re-examined. In this case, equation (6.6) no longer accurately gives the rate of momentum transfer from the electrons to the heavy particles, as is reflected by the term \(-n_e v_{eh}^{(1)} U_e \partial f^0 / \partial C \) in the electron equation (VII 6.40). With respect to the calculation of the heavy-particle transport properties, it turns out that this does not matter, since when \( U_h \gg (m_e/m_h)^{1/2} U_e \) the momentum transfer in question will be small and the term (6.6) can be omitted from the heavy-particle equations. On the other hand, in this case to calculate the electron transport properties the theory of Sec. 4 must now be extended to account for the effects of the heavy-particle diffusion velocities on the electrons.

In many situations it is not necessary in practice to actually evaluate the integral in equation (6.6). This term does not affect the calculation of the viscosity or the heavy-particle thermal-conductivity \( \lambda' \) but only enters in the evaluation of the heavy-particle diffusion velocities and in the thermal-diffusion term in the heavy-particle heat flux (6.2). This is in accord with the
numerical calculations of DeVoto (1967), who evaluated the $\eta$ and $\lambda$ for partially ionized atmospheric-pressure argon, first using the complete formalism of Hirschfelder, Curtiss, and Bird and then neglecting entirely the presence of the electrons [including the term (6.6)] in the evaluation of $\mu$ and the heavy-particle contribution to $\lambda$. Over a range of temperatures from 5000°K to 20,000°K, he found less than a 1% change in both coefficients, when the same Sonine-polynomial approximation is used for each method.

The foregoing discussion has been limited, nominally, to the case of zero magnetic field. Extension of the Chapman-Enskog theory of a multi-component plasma to the magnetic-field case has been worked out by DeVoto (1969) and by Bisnovatyy-Kogan (1964) [see Kalikhman (1967)]. Although this theory already incorporates certain features associated with the small mass of the electron, Daybelge (1968) shows that the results can be further simplified in exactly the same way as in the case of zero magnetic field. Except for very strong magnetic fields or low densities, however, the ratio of the ion cyclotron frequency $eB/m_i$ to the ion collision frequency will be small. When this is so, the simplified equations for the heavy-particle transport properties reduce to those of the zero-magnetic-field case. Accordingly, under these conditions our previous discussion of the use of the results of Hirschfelder, Curtiss, and Bird is directly applicable and the magnetic field enters only in the calculations of Sec. 4 for the electron transport effects and, thereby, in the evaluation of the effective forces (6.6).

Having established the sources that can be used for the calculation of the heavy-particle transport properties, we return in the following section to the consideration of electron transport properties. In particular, with the use of results of Sec. 4, we shall examine the accuracy of various approximate formulas for these properties.

### 7. MIXTURE RULES

On the basis of the theory of Sec. 4 the electron transport properties can be calculated to any accuracy desired by means of the Sonine-polynomial expansions. These calculations incorporate the simplifications associated with the small mass of the electron and hence are much less complicated than the corresponding classical calculations for a gas mixture. None the less, in many practical cases it is desirable or necessary to use for the transport properties approximate formulas that can be evaluated with a minimum of numerical effort. In this section we discuss such formulas, called mixture rules, and assess their accuracy in comparison with the exact theory of Sec. 4. Throughout the section we shall restrict our attention to electron transport properties.