expressed as $\overline{v}_{\text{e}}/\overline{v}_{\text{h}}$ (the abscissa). The approximate boundary on the ordinate between regions of weak and strong fields can be found with the use of equation (3.17). The classical Chapman-Enskog theory of transport properties applies strictly only in the weak-field region where $T_e = T_h$ and $f^0 = f_{\text{M}}(T_h)$. As we shall find in the following section, however, by means of the Cartesian-tensor expansion it is possible to extend the rigorous theory of transport properties to the entire region where $f^0$ is Maxwellian.

**Exercise 3.1.** Show that the condition (3.8) insures that the inequality (3.12) will always be satisfied in those situations for which the energy equation is given by equation (3.17).

**Exercise 3.2.** Derive from equation (3.5) the collision term (3.14) in the electron energy equation (3.16), and show that equation (3.16) follows from equation (VII 3.10c).

**Exercise 3.3.** With the use of the Saha equation (II 10.5b), find the approximate range in temperature and pressure for which the inequality (3.8) is satisfied for equilibrium partially ionized argon.

4. PARTIALLY IONIZED PLASMAS

The Chapman-Enskog theory of the transport properties of a gas mixture is discussed by Chapman and Cowling (1952) and by Hirschfelder, Curtiss and Bird (1964). In this theory it is assumed that the driving mechanisms for translational nonequilibrium of the gas are such that the departures of the distribution functions of the various species from a Maxwellian distribution are small. Such conditions are in fact the microscopic justification for the description of the nonequilibrium phenomena of viscous stress, heat conduction, and diffusion in terms of transport properties. As a consequence of these conditions, it is found in the classical theory that the temperatures of the species remain equal in a nonequilibrium situation. In the preceding section, we have seen that similar limitations on the degree of nonequilibrium [equations (3.10) and (3.12)] are necessary in order that the isotropic part of the electron distribution function in a plasma be Maxwellian. As a result of the small electron mass, however, the electron temperature as given by the electron energy equation can depart significantly from the heavy-particle temperature.

In this section, we consider nonequilibrium situations where the conditions of Sec. 3 apply and discuss the solution for the anisotropic part $f^i$ of the electron distribution function. This solution yields expressions for the electron current density and heat-flux vector in the presence of a magnetic
field in terms of transport coefficients. These coefficients can be calculated from the theory to varying degrees of approximation. On the basis of mean-free-path considerations, the electron viscosity coefficient $\eta_e$, which depends on the solution for $f^2$, is of the order of

$$\frac{\eta_e}{\eta_h} \sim \frac{n_e}{n_h} \left( \frac{m_e T_e}{m_h T_h} \right)^{1/2}$$

when compared with the heavy-particle viscosity $\eta_h$. Since the electron contribution to the viscous stress is therefore almost always negligible, we shall not concern ourselves with the calculation of $\eta_e$.

As we have already indicated, our emphasis on electron transport properties stems from the special role of electrons in plasma transport phenomena, particularly in the presence of a magnetic field, and the particular simplifications that are possible in the calculations of these properties. For example, in many situations the current in a plasma is carried almost entirely by the electrons. Furthermore, as we saw in Chapter VII, the electron Boltzmann equation can be decoupled from the heavy-particle Boltzmann equations, so that the solution for the electron transport properties can be considered separately. The consequences of this decoupling on the calculation of the heavy-particle transport properties will be discussed briefly in Sec. 6.

When the conditions of Sec. 3 are satisfied, the isotropic part $f^0$ of the electron distribution is Maxwellian and the function $f^0$ is governed solely by equation (VII 6.40). We shall not consider at present high-frequency electric fields and, on the basis of the condition $U_h \ll U_e$ discussed in Sec. 2, shall neglect the part of the collision term in equation (VII 6.40) that involves $U_h$. Equation (VII 6.40) with the time derivative omitted then becomes

$$f_{Me} C \left[ \frac{5}{2} - \frac{e^2}{kT_e} \right] \nabla \ln T_e - \frac{e}{kT_e} \mathbf{E} = - \omega_e \times f^0 + v_{ehf} f^0 - \frac{1}{n_e} C_{Me}, \quad (4.1)$$

where $C_{Me}$ is now given by the use of $f^0 = f_{Me}$ in equation (VII 6.36). In writing equation (4.1) we have used the relation [cf. equation (2.32)]

$$\frac{1}{n_e} C \nabla n_e f_{Me} = C f_{Me} \left[ \nabla \ln n_e + \left( \frac{m_e C^2}{2kT_e} - \frac{3}{2} \right) \nabla \ln T_e \right]$$

$$= C f_{Me} \left[ \nabla \ln n + \left( \frac{m_e C^2}{2kT_e} - \frac{5}{2} \right) \nabla \ln T_e \right]$$
Section 4

and the following notation:

\[ \zeta^2 = \frac{m_e c^2}{2kT_e}, \]

\[ \omega_e \equiv eB/m_e, \]

(4.2)

and, as before,

\[ \mathbf{E}' + \frac{\nabla p_e}{e \rho_e} + \frac{m_e}{e} \frac{D \mathbf{u}}{Dt} \]

\[ \approx \mathbf{E}' + \frac{\nabla p_e}{e \rho_e}, \]

(4.3)

As in Sec. 2, the generalized electric field \( \mathbf{E} \) contains in \( \nabla p_e/\rho_e \) a part of the term \( C \nabla n_e f_M \). Again, as indicated, the last term in \( \mathbf{E} \) can be neglected in almost all practical situations.

Equation (4.1) is a linear equation for \( f^1 \), since \( C^1 e e \) is a linear function of \( f^1 \). Furthermore, the driving terms \( \nabla \ln T_e \) and \( \mathbf{E} \) for translational nonequilibrium appear linearly and are mutually independent. Accordingly, the solution of equation (4.1) can be written in the form [cf. equation (2.35)]

\[ f^1 = f_M C(A_1 \nabla \parallel \ln T_e + A_2 \nabla \perp \ln T_e + A_3 \mathbf{b} \times \nabla \ln T_e \]

\[ - (e/kT_e)(D_1 \mathbf{E} \parallel + D_2 \mathbf{E} \perp + D_3 \mathbf{b} \times \mathbf{E}) \],

(4.4)

where we have introduced the factor \(-e/kT_e\) for later convenience. As in Sec. 2, the subscripts \( \parallel \) and \( \perp \) refer to components parallel and perpendicular to the magnetic field. Equations for the unknown scalar functions \( A_1, A_2, A_3, D_1, D_2, \) and \( D_3 \) of electron speed are obtained by substituting equation (4.4) in equation (4.1) and equating coefficients in the three mutually perpendicular directions \( \parallel, \perp, \mathbf{b} \times \) for both \( \nabla \ln T_e \) and \( \mathbf{E} \). Using the fact that \( C^1 e e \) is a linear function of \( f^1 \), we thus obtain the equations

\[ f_M C\left( \frac{5}{2} - \zeta^2 \right) = + v_{eh} f_M C A_1 - C^1 e e[f_M C A_1]/n_e \]

\[ f_M C\left( \frac{5}{2} - \zeta^2 \right) = \omega_e f_M C A_3 + v_{eh} f_M C A_2 - C^1 e e[f_M C A_2]/n_e \]

\[ 0 = -\omega_e f_M C A_2 + v_{eh} f_M C A_3 - C^1 e e[f_M C A_3]/n_e \]

\[ f_M C = + v_{eh} f_M C D_1 - C^1 e e[f_M C D_1]/n_e \]

\[ f_M C = \omega_e f_M C D_3 + v_{eh} f_M C D_2 - C^1 e e[f_M C D_2]/n_e \]

\[ 0 = -\omega_e f_M C D_2 + v_{eh} f_M C D_3 - C^1 e e[f_M C D_3]/n_e. \]

(4.5)
Here \( C^{1}_{ee}[\phi] \) denotes the linear, scalar function of \( \phi \) that is obtained from equation (VII 6.36) for \( C^{1}_{ee} \) by the replacement of \( f^{0} \) by \( f_{M} \) and \( f^{1} \) by \( \phi \) in that equation. That is, \( C^{1}_{ee}[\phi] \) is given by

\[
C^{1}_{ee}[\phi] = n_{e}^{2} \Gamma_{ee} \left[ 8\pi f_{M} \phi + \frac{1}{15C^{2}} \frac{\partial f_{M}}{\partial C} \left( -3l_{1}^{e} + 5l_{2}^{e} + 2J_{-2}^{e} \right) \right. \\
+ \frac{1}{5C} \frac{\partial^{2} f_{M}}{\partial C^{2}} \left( l_{1}^{e} + J_{-2}^{e} \right) + \frac{1}{3C^{2}} \left( -l_{1}^{M} + 3l_{0}^{M} + 2J_{-1}^{M} \right) \left( -\frac{\phi}{C} + \frac{\partial \phi}{\partial C} \right) \left. \\
+ \frac{1}{3C} \left( l_{2}^{M} + J_{-1}^{M} \right) \frac{\partial^{2} \phi}{\partial C^{2}} \right],
\]

(4.6)

where

\[
l^{\phi}_{p} = \frac{4\pi}{C^{p}} \int_{0}^{C} C^{p+2} \phi \, dC,
\]

\[
J^{\phi}_{p} = \frac{4\pi}{C^{p}} \int_{C}^{\infty} C^{p+2} \phi \, dC,
\]

\[
l^{M}_{p} = \frac{4\pi}{C^{p}} \int_{0}^{C} C^{p+2} f_{M} \, dC,
\]

and

\[
J^{M}_{p} = \frac{4\pi}{C^{p}} \int_{C}^{\infty} C^{p+2} f_{M} \, dC.
\]

We note from equations (4.5) that the equations for \( A_{1} \) and \( D_{1} \) are independent of \( \omega_{e} \), or equivalently \( B \), and that the equations for \( A_{2} \) and \( D_{2} \) reduce to those for \( A_{1} \) and \( D_{1} \) when \( B = 0 \). It follows that \( A_{1} = A_{2}(B = 0) \) and that \( D_{1} = D_{2}(B = 0) \). We can therefore limit our consideration to the solutions for \( A_{2}, A_{3}, D_{2}, \) and \( D_{3} \) and recover \( A_{1} \) and \( D_{1} \) by setting \( B = 0 \) in the solutions for \( A_{2} \) and \( D_{2} \). The solutions for these quantities can be most conveniently obtained in terms of the complex variables \( A \) and \( D \) that are defined by

\[
A \equiv A_{2} + iA_{3}
\]

and

\[
D \equiv D_{2} + iD_{3}
\]

where \( i^{2} = -1 \). Equations (4.5) then reduce to

\[
f_{M} \phi(\frac{3}{2} - \phi^{2}) = (v_{eH} - i\omega_{e}) f_{M} \phi A - C^{1}_{ee}[f_{M} \phi A] / n_{e}
\]

(4.8)

and

\[
f_{M} \phi = (v_{eH} - i\omega_{e}) f_{M} \phi D - C^{1}_{ee}[f_{M} \phi D] / n_{e}.
\]

(4.9)
The solutions of Sec. 2 for a Lorentzian plasma, with \( f^0 = f^M \), can be recovered directly from equations (4.8) and (4.9) by setting \( C_{C\varepsilon} \) equal to zero in these equations.

Before proceeding with the solution of equations (4.8) and (4.9) for \( A \) and \( D \), we examine the consequences of the use of the form (4.4) for \( f' \) in the expressions (2.12) and (2.31) for the electron current density \( J_e \) and heat flux \( q_e \). These expressions are

\[
J_e = -en_e \frac{4\pi}{3} \int_0^\infty C^3 f^1 \, dC
\]

and

\[
q_e = n_e m_e \frac{2\pi}{3} \int_0^\infty C^3 f^1 \, dC = -\frac{5kT_e}{2e} J_e + n_e m_e \frac{2\pi}{3} \int_0^\infty \left( C^2 - \frac{5kT_e}{m_e} \right) C^3 f^1 \, dC,
\]

where we have written the second form for \( q_e \) to simplify our later calculations. With the use of equation (4.4) for \( f' \), \( J_e \) and \( q_e \) can be written directly in terms of transport coefficients as

\[
J_e = \sigma_\parallel \mathbf{E}_\parallel + \sigma_\perp \mathbf{E}_\perp + \sigma_H \mathbf{b} \times \mathbf{E} + \phi_\parallel \nabla \parallel T_e + \phi_\perp \nabla \perp T_e + \phi_H \mathbf{b} \times \nabla T_e, \tag{4.11}
\]

and

\[
q_e = -\frac{5kT_e}{2e} J_e - \lambda_\parallel \nabla \parallel T_e - \lambda_\perp \nabla \perp T_e - \lambda_H \mathbf{b} \times \nabla T_e
\]

\[
- T_e \phi_\parallel \mathbf{E}_\parallel - T_e \phi_\perp \mathbf{E}_\perp - T_e \phi_H \mathbf{b} \times \mathbf{E}. \tag{4.12}
\]

The coefficients of electrical conductivity, \( \sigma \), thermal conductivity, \( \lambda' \), and thermal diffusion \( \phi \) and \( \phi' \) are given by

\[
\sigma_\parallel = \frac{n_e e^2}{kT_e} \frac{4\pi}{3} \int_0^\infty C^4 f_m \, dC = \frac{n_e e^2}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \mathcal{G}_4 D_1 e^{-\mathcal{G}} d\mathcal{G},
\]

\[
\sigma_\perp = \frac{n_e e^2}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \mathcal{G}_4 D_2 e^{-\mathcal{G}} d\mathcal{G},
\]

\[
\sigma_H = \frac{n_e e^2}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \mathcal{G}_4 D_3 e^{-\mathcal{G}} d\mathcal{G},
\]

\[
\phi_\parallel = -\frac{n_e e k}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \mathcal{G}_4 A_1 e^{-\mathcal{G}} d\mathcal{G},
\]
\[
\phi_\parallel = -\frac{n_e e k}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \varphi^4 A_2 e^{-\varphi^2} \, d\varphi
\]
\[
\phi_H = -\frac{n_e e k}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \varphi^4 A_3 e^{-\varphi^2} \, d\varphi,
\]
\[
\lambda_\parallel = -\frac{n_e k^2 T_e}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \left( \varphi^2 - \frac{5}{2} \right) \varphi^4 A_1 e^{-\varphi^2} \, d\varphi,
\]
\[
\lambda_\perp = -\frac{n_e k^2 T_e}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \left( \varphi^2 - \frac{5}{2} \right) \varphi^4 A_2 e^{-\varphi^2} \, d\varphi,
\]
\[
\lambda_H = -\frac{n_e k^2 T_e}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \left( \varphi^2 - \frac{5}{2} \right) \varphi^4 A_3 e^{-\varphi^2} \, d\varphi,
\]
\[
\phi_{\parallel}' = \frac{n_e e k}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \left( \varphi^2 - \frac{5}{2} \right) \varphi^4 D_1 e^{-\varphi^2} \, d\varphi,
\]
\[
\phi_{\perp}' = \frac{n_e e k}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \left( \varphi^2 - \frac{5}{2} \right) \varphi^4 D_2 e^{-\varphi^2} \, d\varphi,
\]
\[
\phi_H' = \frac{n_e e k}{m_e} \frac{8}{3\sqrt{\pi}} \int_0^\infty \left( \varphi^2 - \frac{5}{2} \right) \varphi^4 D_3 e^{-\varphi^2} \, d\varphi.
\]

Although the expressions (4.13) when taken together are complicated in appearance, their derivation from equations (4.4) and (4.10) is completely straightforward. As in Sec. 2, the subscript \( H \) denotes the Hall direction, which is mutually perpendicular to the magnetic field and to the driving "force," \( \mathbf{E} \) or \( \nabla T_e \). We shall find from the following analysis that the two thermal-diffusion coefficients \( \phi \) and \( \phi' \) are actually the same; that is
\[
\phi_{\parallel}' = \phi_{\parallel}
\]
\[
\phi_{\perp}' = \phi_{\perp}
\]
\[
\phi_H' = \phi_H.
\]

The coefficients \( \lambda' \) differ from the usual coefficients of thermal conductivity \( \lambda \). The coefficients \( \lambda \) are such that the heat flux is proportional to \( \lambda \) when the current density is zero. In terms of \( \lambda \), equation (4.12) for the heat flux can be rewritten
\[
\mathbf{q}_e = -\lambda_{\parallel} \nabla_{\parallel} T_e - \lambda_{\perp} \nabla_{\perp} T_e - \lambda_H \mathbf{b} \times \nabla T_e + \psi_{\parallel} \mathbf{J}_{e\parallel} + \psi_{\perp} \mathbf{J}_{e\perp} + \psi_H \mathbf{b} \times \mathbf{J}_e,
\]
\[
(4.14)
\]
where the \( \psi \)'s are new coefficients.
The coefficients $\lambda$ and $\psi$ can be found with the aid of equation (4.11). The result is (see Marshall, 1960)

\[
\lambda_{||} = \lambda_{||}^* - T_e \frac{\phi_{||}}{\sigma_{||}}
\]

\[
\lambda_{\perp} = \lambda_{\perp}^* - T_e \frac{\sigma_{\perp}(\phi_{\perp}^2 - \phi_{||}^2) + 2\sigma_{H} \phi_{\perp} \phi_{H}}{\sigma_{\perp}^2 + \sigma_H^2},
\]

\[
\lambda_{H} = \lambda_{H}^* - T_e \frac{\sigma_{H}(\phi_{H}^2 - \phi_{||}^2) + 2\sigma_{\perp} \phi_{\perp} \phi_{H}}{\sigma_{\perp}^2 + \sigma_H^2},
\]

(4.15)

\[
\psi_{||} = -\frac{5}{2} \frac{k T_e}{e} - \frac{T_e}{\sigma_{||}} \phi_{||},
\]

\[
\psi_{\perp} = -\frac{5}{2} \frac{k T_e}{e} - \frac{T_e}{\sigma_{\perp}} \phi_{\perp} + \frac{\sigma_{H} \phi_{H}}{\sigma_{\perp}^2 + \sigma_H^2},
\]

and

\[
\psi_{H} = - \frac{T_e}{\sigma_{\perp}} \phi_{H} - \frac{\sigma_{H} \phi_{H}}{\sigma_{\perp}^2 + \sigma_H^2}.
\]

Here we have used the previously mentioned fact that $\phi^* = \phi$.

It is also possible to express the heat flux $q_e$ directly in terms of $\nabla T_e$ and $\delta$, with the elimination of $J_e$ from equation (4.12) by means of equation (4.11). This is the form that we used in Sec. 2, with the coefficients of $\nabla T_e$ in that expression denoted by $\lambda'$ [see equation (2.38)]. Our present choice of the form (4.12) for $q_e$ is motivated by the fact that for a partially ionized plasma the coefficients $\sigma, \lambda'$, and $\phi$ can be calculated most readily. Once these quantities are known, the coefficients $\lambda$ and $\psi$ can be found from equations (4.15). In the absence of a magnetic field, the foregoing relations reduce to the following forms:

\[
J_e = \sigma_{||} \delta + \phi_{||} \nabla T_e
\]

and

\[
q_e = -\frac{5}{2} \frac{k T_e}{e} J_e - \lambda_{||} \nabla T_e - \frac{T_e}{\sigma_{||}} \phi_{||} \delta
\]

\[
= - \left(\lambda_{||} + \frac{5}{2} \frac{k T_e}{e} \phi_{||}\right) \nabla T_e - \frac{T_e}{\sigma_{||}} \left(\phi_{||} + \frac{5}{2} \frac{k}{e} \phi_{||}\right) \delta
\]

(4.16)

or

\[
q_e = - \lambda_{||} \nabla T_e + \psi_{||} J_e.
\]
At this point we have established the functional dependence of the fluxes \( J_e \) and \( q_e \) on the driving forces \( \mathcal{F} \) and \( \nabla T_e \). We now consider the calculation of the transport properties themselves. As in the kinetic theory of gas mixtures, although a closed-form solution for these properties is not possible, an approximate solution can be obtained by means of expansion in terms of Sonine polynomials. The Sonine polynomials \( S_n^m \) are defined by

\[
S_n^m[x] = \sum_{p=0}^{n} \frac{(m + n)!}{(m + p)! p! (n - p)!} (-x)^p
\]

so that, for example,

\[
S_n^0[x] = 1
\]

and

\[
S_n^1[x] = m + 1 - x.
\]

These polynomials have the orthogonality property

\[
\int_0^\infty x^n e^{-x} S_n^m(x) S_q^p(x) \, dx = 0 \quad \text{when } n \neq q
\]

\[
= \frac{(m + n)!}{n!} \quad \text{when } n = q.
\]

In this relation the factorial of a number other than an integer is to be interpreted as a \( \Gamma \)-function. In particular, if \( q \) is an integer, we have

\[
\left( \frac{q}{2} \right)! = \left( \frac{q}{2} \right) \left( \frac{q}{2} - 1 \right) \cdots \frac{1}{2} \sqrt{\pi}.
\]

To obtain approximate solutions to equations (4.8) and (4.9) for the quantities \( A \) and \( D \), we write

\[
A \approx A^{(q)} \equiv \sum_{r=1}^{\xi} \alpha_r^{(q)} S_{3/2}^{-1} [\mathcal{F}^2]
\]

and

\[
D \approx D^{(q)} \equiv \sum_{r=1}^{\xi} \delta_r^{(q)} S_{3/2}^{-1} [\mathcal{F}^2].
\]

The expansion coefficients \( \alpha_r^{(q)} \) and \( \delta_r^{(q)} \) depend on the order \( \xi \) of the approximation. Since the quantities \( A \) and \( D \) are complex [see equations
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(4.7)], \(\sigma^{(q)}_f\) and \(\delta^{(q)}_f\) are also complex. The transport properties \(\sigma_f\), \(\lambda_f\), and \(\phi\) can be obtained in terms of the expansion coefficients \(\sigma^{(q)}_f\) and \(\delta^{(q)}_f\) by substitution of the expansions (4.20) and (4.21) in equations (4.13). With the use of the orthogonality property (4.18) and equations (4.17), we find that

\[
\sigma^{(q)}_f + i\sigma^{(q)}_f = \frac{n\epsilon e^2}{m_e} \delta^{(q)},
\]

\[
\phi^{(q)}_f + i\phi^{(q)}_f = -\frac{n\epsilon e k}{m_e} \sigma^{(q)},
\]

\[
\lambda^{(q)}_f + i\lambda^{(q)}_f = \frac{5n\epsilon k^2 T_e}{2m_e}-\sigma^{(q)},
\]

(4.22)

and

\[
\phi^{(q)}_f + i\phi^{(q)}_f = -\frac{5n\epsilon e k}{2m_e} \delta^{(q)}.
\]

(The proof that \(\phi' = \phi\), which follows, is based on these relations.) Since \(A_1 = A_2(B = 0)\) and \(D_1 = D_2(B = 0)\), we have also

\[
\sigma^{(q)}_f = \sigma^{(q)}_f(B = 0),
\]

\[
\phi^{(q)}_f = \phi^{(q)}_f(B = 0),
\]

\[
\lambda^{(q)}_f = \lambda^{(q)}_f(B = 0),
\]

\[
\phi^{(q)}_f = \phi^{(q)}_f(B = 0).
\]

(4.23)

The expansion coefficients are subject to the governing equations (4.8) and (4.9) for \(A\) and \(D\). These equations can be written in terms of Sonine polynomials as

\[
\left(\frac{m_e}{2\pi k T_e}\right)^{3/2} e^{-\frac{q^2}{2} S^{(q)}_y \bar{\delta}^2} = (v_{efh} - i\omega_e) \theta f_f A - C_{ed} [\theta f_f A] / n_e
\]

and

\[
\left(\frac{m_e}{2\pi k T_e}\right)^{3/2} e^{-\frac{q^2}{2} S^{(q)}_y \bar{\delta}^2} = (v_{efh} - i\omega_e) \theta f_f D - C_{ed} [\theta f_f D] / n_e.
\]

(4.24)

The approximate solutions to equations (4.24) are obtained from moment equations that are formed by the substitution in these equations of the
expansions (4.20) and (4.21) for $A$ and $D$, followed by the successive
multiplication of the resulting equations by $\Phi S_{3/2}^{-1}(\hat{\rho}^2) \pi C^2 \, dC$ ($s = 1, 2, 3, \ldots, \xi$) and integration over $C$. These integrations are readily performed
with the use of the orthogonality property (4.18) and yield the following
set of algebraic equations for the expansion coefficients $A_s^{(0)}$ and $\delta_s^{(0)}$:

$$
\sum_{r=1}^{\xi} A_s^{(0)} a_{rs} = \begin{cases} 
\frac{1}{4} \frac{1}{s^2} & \text{if } s = 2 \\
0 & \text{if } s \neq 2 
\end{cases} \quad (4.25)
$$

and

$$
\sum_{r=1}^{\xi} \delta_s^{(0)} a_{rs} = \begin{cases} 
\frac{1}{2} & \text{if } s = 1 \\
0 & \text{if } s \neq 1 
\end{cases} \quad (4.26)
$$

Since $s = 1, 2, 3, \ldots, \xi$, equations (4.25) and (4.26) represent $\xi$ equations
for each of the $\xi$ unknown coefficients $A_s^{(0)}$ and $\delta_s^{(0)}$. The quantities $a_{rs}$ are defined by the relations

$$
a_{rs} = -i\omega_e \frac{2}{\sqrt{\pi}} \frac{(r + \frac{1}{2})!}{(r - 1)!} \delta_{rs} + a_{rs}^{(h)} + a_{rs}^{(e)}, \quad (4.27)
$$

and

$$
a_{rs}^{(h)} = \int_0^{\infty} \Phi^2 S_{3/2}^{-1}(\hat{\rho}^2) S_{3/2}^{-1}(\hat{\rho}^2) \pi C^2 \, dC, \quad (4.28)
$$

$$
a_{rs}^{(e)} = -\frac{1}{n_e} \int_0^{\infty} \Phi^2 S_{3/2}^{-1}(\hat{\rho}^2) \pi C^2 \, dC, \quad (4.29)
$$

where $\delta_{rs}$ is the Kronecker delta.

The definite integrals (4.29) for $a_{rs}^{(e)}$ involve only the electron-electron
collision frequency and hence can be evaluated once and for all. The
pertinent calculations have been performed by several authors (e.g.,
Landshoff, 1949). The results can be expressed in the form

$$
da_{rs}^{(e)} = \frac{3\sqrt{2}}{4} \bar{\nu}_{ee} a_{rs}', \quad (4.30)
$$

in terms of the mean electron-electron collision frequency (2.2b)

$$
\bar{\nu}_{ee} = n_e \frac{8\sqrt{\pi}}{3} \left( \frac{m_e}{kT_e} \right)^{3/2} \left( \frac{e^2}{4\pi\varepsilon_0 m_e} \right)^2 \ln \Lambda. \quad (4.31)
$$
The numbers \( a_{rs} = a_{sr} \) are found to satisfy the identity

\[
- \sum_{r,s} \zeta^{r-1} \eta^{s-1} a_{rs} = \frac{\zeta \eta (-8 + 4\zeta + 4\eta + \zeta \eta - 2\zeta^2 \eta - 2\eta^2 + 3\zeta^2 \eta^2)}{(1 - \zeta \eta)^2(2 - \zeta - \eta)^{5/2}}. \tag{4.32}
\]

That is, the numbers \( a_{rs} \) are equal to the expansion coefficients of the expression on the right-hand side of equation (4.32). Daybelge (1968) has evaluated these quantities up to \( r = s = 12 \). The first few values are

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.75000 & 0.46875 & 0.27344 & 0 & 0 \\
0 & 2.8125 & 2.41406 & 1.72852 & 0 & 0 \\
0 & 5.52441 & 5.52441 & 4.96802 & 0 & 0 \\
0 & 9.13995 & 9.13995 & 8.60448 & 0 & 0 \\
\end{pmatrix}
\]

(4.33)

These values have been calculated on the basis of a "cut-off" Coulomb potential and hence apply when \( \ln \Lambda \gg 1 \) (see Chapter VII, Sec. 4). For smaller \( \ln \Lambda \), Daybelge (1968) and (1970) gives corrections based on the unified theory of Kihara and Aona (1967). From the standpoint of practical calculations, however, the correction for smaller \( \ln \Lambda \) may be most readily applied by means of the Frost mixture rule discussed in Sec. 7.

The integrals \( a_{rs}^{(0)} \) depend on the collision frequency \( v_{eH} = \sum_k v_{ek}^{(1)} \) and must be evaluated for the particular plasma conditions in question. The contributions of the various species to \( a_{rs}^{(0)} \), when divided by \( n_k \), are identical to the small-electron-mass approximations to the so-called bracket integrals that appear in the Chapman-Enskog theory. In the present case, however, the considerations of the dynamics of a collision have already been incorporated in \( a_{rs}^{(0)} \). Thus the evaluation of the integrals \( a_{rs}^{(0)} \) from equation (4.28) and given collision cross sections reduces to a matter of straightforward quadratures.

Once the quantities \( a_{rs} \) have been calculated, the transport properties \( \sigma \), \( \chi \), and \( \phi \) can be found from the solutions of the equations (4.25) and (4.26) for \( \delta_{i}^{(0)}, \delta_{s}^{(0)}, \delta_{2}^{(0)}, \) and \( \delta_{2}^{(0)} \) by the use of equations (4.22) and (4.23). The solutions to equations (4.25) and (4.26) can be expressed in terms of determinants by means of Cramer's rule. The values of the transport properties calculated from these equations will depend on the order \( \xi \) of the approximation and correspond directly to the \( \xi \)th approximations in the classical Chapman-Enskog theory. It follows from equations (4.22), (4.23), (4.25), and (4.26) that nonzero values of \( \sigma \) are obtained for \( \xi \geq 1 \), whereas the calculation of \( \chi \) and \( \phi \) requires \( \xi \geq 2 \).

To prove that the two thermal diffusion coefficients \( \phi' \) and \( \phi \) are
actually the same, we write the solutions to equations (4.25) and (4.26) in terms of determinants as

\[
\begin{vmatrix}
0 & a_{21} & \cdots & a_{21} \\
15 & 4 & a_{22} & \cdots & a_{22} \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & a_{2\xi} & \cdots & a_{2\xi} \\
\end{vmatrix}
\]

\[
= \frac{15}{4} \begin{vmatrix}
a_{21} & a_{31} & \cdots & a_{31} \\
a_{23} & a_{33} & \cdots & a_{33} \\
\cdots & \cdots & \cdots & \cdots \\
a_{2\xi} & \cdots & \cdots & a_{2\xi} \\
\end{vmatrix}
\]

and

\[
\begin{vmatrix}
a_{11} & \frac{3}{2} & a_{31} & \cdots & a_{31} \\
a_{12} & 0 & a_{32} & \cdots & a_{32} \\
0 & \cdots & \cdots & \cdots & \cdots \\
a_{1\xi} & 0 & a_{3\xi} & \cdots & a_{3\xi} \\
\end{vmatrix}
\]

\[
= \frac{3}{2} \begin{vmatrix}
a_{12} & a_{32} & \cdots & a_{32} \\
a_{13} & a_{33} & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
a_{1\xi} & \cdots & \cdots & a_{3\xi} \\
\end{vmatrix}
\]

where \( \Delta \) is the determinant of the coefficients

\[
\Delta = \begin{vmatrix}
a_{11} & a_{21} & \cdots & a_{21} \\
a_{12} & a_{22} & \cdots & a_{22} \\
\cdots & \cdots & \cdots & \cdots \\
a_{1\xi} & \cdots & \cdots & a_{2\xi} \\
\end{vmatrix}
\]

Since \( a_{ss} = a_{ss} \) and since the value of a determinant is unchanged by interchange of the rows and columns, it follows that

\[
\frac{3}{2} \delta_2^{(0)} = \delta_1^{(0)},
\]

for any order \( \xi \) of approximation. Therefore, from equations (4.22) and (4.23) we have for each component \( \phi' = \phi \), as we have previously indicated.

The preceding description of electron transport phenomena in partially ionized plasmas applies whether or not the electron temperature \( T_e \) is equal to the heavy-particle temperature \( T \), as long as the conditions of Sec. 3 are satisfied. Our results are in fact consistent with those obtained from the Chapman-Enskog theory by formal replacement of the gas temperature by the electron temperature. The present formulas are, however, much simpler in that they incorporate explicitly the simplifications that result from the
small electron mass. Accordingly, they provide a simplified basis for the
calculation of the electron contribution to the transport properties when
\( T_e = T_h \), as well as when \( T_e \neq T_h \).

In practice evaluation of the transport properties from the foregoing
equations requires a knowledge of the pertinent momentum-transfer collision
frequencies \( \nu_{\text{eff}}^{(i)} \) and hence cross-sections \( Q_{\text{eff}}^{(i)} \) for the constituents of
the plasma in question. These cross sections can be obtained from a variety
of sources. Representative data are given in Sec. 14 of Chapter II and an
extensive compilation is provided by Brown (1967).

The extension of the foregoing analysis to the calculation of the high-
frequency electrical conductivity of a partially ionized plasma follows closely
the corresponding calculations of Sec. 2. Thus we write [cf. equation (2.24)
and (4.4)]

\[
f^I = \text{Re} \left[ -\frac{e}{kT_e} f_M C (\tilde{D}_1 E_\parallel + \tilde{D}_2 E_\perp + \tilde{D}_3 \mathbf{b} \times \mathbf{E}) e^{-i\omega t} \right]
\] (4.35)

for the part of \( f^I \) that results from a high-frequency field

\[
E^I = \text{Re} e^{-i\omega t}.
\] (4.36)

From equation (4.1), with the term \(-\partial f^I / \partial t\) restored to the left-hand side,
it follows that the complex coefficients \( \tilde{D}_1 \), \( \tilde{D}_2 \), and \( \tilde{D}_3 \) must satisfy the
equations [cf. equations (4.5)]

\[
f_M C = \frac{\omega_e f_M C^{\dagger} - C_{\text{el}} f_M C \tilde{D}_1 f_M C^{\dagger}}{n_e} + (-i\omega + v_{\text{eh}}) f_M C \tilde{D}_1 f_M C^{\dagger} - C_{\text{el}} f_M C \tilde{D}_1 f_M C^{\dagger}
\] (4.37)

and

\[
0 = -\omega_e f_M C \tilde{D}_2 + (-i\omega + v_{\text{eh}}) f_M C \tilde{D}_2 - C_{\text{el}} f_M C \tilde{D}_2 f_M C^{\dagger}
\]

With a certain amount of complex algebra (see Exercise 4.5), it is possible
to show from equations (4.37) that the conclusions of Sec. 2 with regard
to the high-frequency conductivity remain valid. That is, as a result of the
high-frequency field (4.36), there is added to the steady current density (4.11)
a current density given by

\[
J_e = \text{Re} \left[ (\tilde{\sigma}_\parallel E_\parallel + \tilde{\sigma}_\perp E_\perp + \tilde{\sigma}_H \mathbf{b} \times \mathbf{E}) e^{-i\omega t} \right],
\] (4.38)

where the complex conductivities \( \tilde{\sigma} \) satisfy the relations (2.27); that is,

\[
\tilde{\sigma}_\parallel = \sigma_\parallel(\omega) + i\sigma_H(\omega),
\]

\[
\tilde{\sigma}_\perp = \frac{1}{2}[\sigma_\perp(\omega + \omega_e) + \sigma_\perp(\omega - \omega_e)] + i[\sigma_H(\omega + \omega_e) + \sigma_H(\omega - \omega_e)]
\]

and

\[
\tilde{\sigma}_H = \frac{1}{2}[\sigma_H(\omega + \omega_e) - \sigma_H(\omega - \omega_e)] - i[\sigma_\perp(\omega + \omega_e) - \sigma_\perp(\omega - \omega_e)]
\] (4.39)
For a partially ionized plasma the conductivities $\sigma_1$ and $\sigma_H$ in equations (4.39) are now to be evaluated by use of the methods of this section.

In the Chapman-Enskog calculation of the transport properties of a mixture of neutral gases it is found that in most cases the convergence of the Sonine polynomial expansions is rapid. Usually only a few terms in the series which correspond to our equations (4.20) and (4.21) are required to obtain accurate values of the various transport properties. The convergence of these expansions for partially ionized plasmas may be significantly slower, however, primarily as a result of the behavior of certain cross sections for electron-neutral collisions. Generally speaking, for plasmas the convergence is less favorable when the electron-neutral cross section in the vicinity of the electron thermal speed increases rapidly with increasing electron speed and when the degree of ionization is low.

We consider first the convergence of the successive approximations for the electron transport properties in the Lorentzian limit of a weakly ionized plasma, for which the exact values of the transport properties are known from the results of Sec. 2. In Fig. 4 the convergence of the electrical and thermal conductivities $\sigma_1$ and $\lambda$ is shown for several power-law models of the electron/heavy-particle collision frequency, for weakly ionized argon at $T_e = 4000^\circ K$, and for weakly ionized cesium at $T_e = 1500^\circ K$. For the

![Figure 4: Convergence in the Lorentzian limit of the successive Sonine-polynomial values $\sigma_1$ and $\lambda$ as compared to the exact values. To emphasize the trends, the discrete points $\xi = 1, 2, 3, ...$ are connected by curves.](image-url)
power-law models the speed dependence of the momentum-transfer cross section \( Q_{\text{eff}}^{(1)}(C) \) is represented in terms of a number \( m \) by the proportionality
\[
Q_{\text{eff}}^{(1)}(C) \propto C^{2m-1}.
\]
For a single heavy-particle species the collision frequency \( v_{\text{eff}} \) is then of the form
\[
v_{\text{eff}} = n_h C Q_{\text{eff}}^{(1)} \propto C^{2m}.
\] (4.40)
The values of \( \sigma^{(0)}_\parallel \) and \( \lambda^{(0)}_\parallel \) used to obtain Fig. 4 have been calculated from equations (4.22), (4.23), and (4.25) through (4.28) with \( \sigma^{(e)}_\parallel \) and \( \omega_e \) set equal to zero. The exact values in the Lorentzian limit are given by equation (2.14a), with \( f^0 = f_M \), for \( \sigma_\parallel \) and the use of the results of Sec. 2 in the formula \( \lambda_\parallel = \tilde{\lambda}_\parallel - T_e \phi_\parallel (5k/2e + \phi_\parallel/\sigma_\parallel) \) (see Exercise 4.2). For the model cross sections it is found that \( \sigma^{(0)}_\parallel/\sigma^{(\text{exact})}_\parallel = \lambda^{(0)}_\parallel/\lambda^{(\text{exact})}_\parallel \); for the argon and cesium this does not hold and only \( \sigma^{(0)}_\parallel/\sigma^{(\text{exact})}_\parallel \) is shown. It can be seen that for increasing values of \( m \) and for argon the convergence is less rapid. We note that \( m = -\frac{3}{2} \) corresponds to Coulomb collisions and that \( m = \frac{1}{2} \) represents a constant, "hard-sphere" cross section which is independent of \( C \).

In the case of argon, the pronounced Ramsauer minimum in the cross section at low electron energies has an adverse effect similar to a uniformly increasing cross section (see Chapter II, Fig. 25 which gives the argon cross section measured by Golden used to prepare Figs. 4 and 5).

Of the usual constituents of partially ionized plasmas, argon can be regarded as the least favorable with regard to the convergence of the successive approximations for the transport properties. In fact, Fig. 4 shows that even the twelfth approximation does not give an accurate value for \( \sigma_\parallel \) of argon in the Lorentzian limit. On the other hand, the convergence is much more rapid in the case of cesium, for which the cross section—shown in Chapter II, Fig. 22—is relatively insensitive to electron energy at low energies.

Consistent with Fig. 4, the principal observations which we wish to make with regard to convergence are as follows: In certain cases, such as weakly ionized argon, approximations of a high order are required for accurate calculations. In most cases, however, for most plasma constituents, as for the weakly ionized cesium, the second or third approximation is satisfactory for practical calculations. Furthermore, the error incurred by the use of an approximation of low order is usually no more than approximately a factor of two.

Since as shown in Fig. 4 convergence is good for \( m = -\frac{3}{2} \), which corresponds to Coulomb collisions, it would be expected that the convergence would become more rapid as the degree of ionization is increased. This is
confirmed by Fig. 5 which shows the successive approximations for $\sigma_\parallel^{(0)}$, $\lambda_\parallel^{(0)}$, and $\phi_\parallel^{(0)}$ as a function of temperature for atmospheric-pressure equilibrium argon for which $T_e = T_s$. On the abscissa the degree of ionization, based on the use of the Saha equation (II 10.5b), is expressed as $\bar{\nu}_e/\bar{\nu}_{el}$ [see equation (3.15)]. Here the values of the transport properties given by the twelfth approximation can be regarded as exact, except for the lowest degrees of ionization (cf. Fig. 4).

The convergence of the successive approximations for the thermal conductivity $\lambda_\parallel$ and the thermal diffusion coefficient $\phi_\parallel$ is similar to that for $\sigma_\parallel$. The change in sign of $\phi_\parallel$ as the degree of ionization increases is typical of the behavior of real plasmas. On the right in Fig. 5 the values of the transport coefficients for fully ionized plasmas are indicated. For example, for a fully ionized plasma $\sigma_\parallel^{(2)} = 0.978 \sigma_\parallel^{(exact)}$. (The calculation of the exact value is discussed in the following section.)

The convergence of the perpendicular and Hall coefficients of electrical conductivity is illustrated in Fig. 6. Again it is convenient to consider the Lorentzian limit of Sec. 2 where the exact values can be easily calculated. In this figure $\sigma_\perp^{(0)/\sigma_\perp^{(exact)}}$ and $\sigma_H^{(0)/\sigma_H^{(exact)}}$ are shown as functions of magnetic field expressed in terms of $\beta_e = \omega_e/\sqrt{\nu}_{el}$, for the case of constant cross sections for which $\bar{\nu}_e \propto C$. Since $\sigma_\perp (B = 0) = \sigma_H$, the convergence of $\sigma_\perp$ for small $\beta_e$ is the same as that given in Fig. 4 for $m = \frac{1}{2}$. In contrast to results of Figs. 4 and 5 for $\sigma_\parallel$, the convergence at some magnetic fields is not monotonic. (For the present example this is apparent from Fig. 6 only for $\sigma_\perp$, although for other cross sections a clearly nonmonotonic convergence at some fields is observed for $\sigma_H$ as well.) We note here for later reference that the curves for $\sigma_\perp^{(1)}$ and $\sigma_H^{(1)}$ show the accuracy of the mean-free-path formulas for the perpendicular and Hall conductivities, since it is readily demonstrated that

$$\sigma_\perp^{(1)} = \frac{n_e e^2/m_e \bar{\nu}_{el}}{1 + \beta_e^2},$$

and

$$\sigma_H^{(1)} = \frac{(n_e e^2/m_e \bar{\nu}_{el}) \beta_e}{1 + \beta_e^2},$$

where $\beta_e = \omega_e/\sqrt{\nu}_{el}$. These results apply for an arbitrary degree of ionization (see Exercise 4.3).

The convergence of $\sigma_\perp$ and $\sigma_H$ in the alternate limit of a fully ionized plasma is shown in Fig. 7 for the case of singly charged ions. Here it is appropriate to base the normalizations on $\beta_e = \omega_e/\sqrt{\nu}_{el}$ and the exact value of $\sigma_\parallel$ for a fully ionized plasma, which is given in the following section.
Figure 5. Convergence of the successive approximations for $\sigma_\parallel$, $\lambda_\parallel$, and $\phi_\parallel$ for atmospheric-pressure argon.
Figure 6. The convergence of $\sigma_p$ and $\sigma_H$ in the Lorentzian limit as a function of magnetic field for the case of constant cross sections for which $v_{th} \propto C$.

The results given in Figs. 6 and 7 can be taken as reasonably representative of the convergence of the perpendicular and Hall components of the various transport properties in the presence of a magnetic field, with the exception of the somewhat complicated sign behavior of the thermal diffusion coefficient which we have noted in Sec. 2 (see also Fig. 9 in the following section). The relatively slow convergence shown in Fig. 4 for cross sections that are increasing functions of electron energy applies to the perpendicular and Hall components as well, for low values of $\beta_e$. For higher values of $\beta_e$, the convergence is more rapid, in accord with the results of Figs. 6 and 7.
Exercise 4.1. Write the coefficients in the expression for the heat flux $q_e$ in terms of $V T_e$ and $\mathcal{S}$ as functions of the parallel, perpendicular, and Hall components of $\sigma, \lambda', \phi$.

Exercise 4.2. 1. Derive the formula $\lambda_\parallel = \lambda_\parallel \text{ (exact)} - T_e \phi (5k/2e + \phi_\parallel / \sigma_\parallel)$. 2. Obtain an integral expression for the Lorentzian value of $\lambda_\parallel$ [cf. equation (7.5a), which follows].

Exercise 4.3. Find expressions for the first nonzero approximations for the parallel, perpendicular, and Hall components of $\sigma, \lambda', \lambda$, and $\phi$. Express
Exercise 4.4. Show, by means of estimates of the magnitude of the electrical conductivity and thermal-diffusion coefficients, that the thermal-diffusion current produced by a temperature gradient $\nabla k T_e e^2$, measured in eV/cm, is similar to that produced by a comparable electric field, measured in V/cm ($1 \text{ eV} \approx 11,600 \text{ eV}$). For simplicity, take $B = 0$. Similarly, estimate the conditions for which terms other than those that involve $\nabla T_e$ will be important in equations (4.16) for the heat flux.

Exercise 4.5. Demonstrate the validity of equations (4.39) for the complex conductivities $\bar{\sigma}_\parallel$, $\bar{\sigma}_\perp$, and $\bar{\sigma}_H$.

5. FULLY IONIZED PLASMAS

A fully ionized plasma is described as a special case by the equations of the previous section. Since for a fully ionized plasma the number density of neutral particles vanishes, the heavy-particle collision frequency $v_{ei}$ becomes $v_{ei} = \sum_i n_i C Q_{ei}^{(i)}$, where $Q_{ei}^{(i)}$ is given by equation (VII 4.10). Here the summation reflects the fact that, in general, there can be more than one species of ions. With this substitution, the anisotropic part of the electron distribution function is governed by equations (4.4), (4.8), and (4.9) as before.

In the case where the magnetic field is zero the integro-differential equations (4.8) and (4.9) have been solved for fully ionized plasmas by Spitzer and Härm (1953) by means of numerical integration. For singly charged ions their results yield the following values for the electron transport coefficients:

\[
\sigma_\parallel = 0.582 \times \frac{32 \sqrt{\pi} e^3 (2kT_e)^{3/2}}{e^2 \sqrt{m_e \ln \Lambda}} = 1.975 \frac{n_e e^2}{m_e \tilde{v}_{ei}} \tag{5.1a}
\]

\[
\lambda_\parallel = 0.236 \times \frac{64 \sqrt{\pi} e^5 k (2kT_e)^{5/2}}{e^4 \sqrt{m_e \ln \Lambda}} = 3.203 \frac{n_e k^2 T_e}{m_e \tilde{v}_{ei}} \tag{5.1b}
\]

\[
\phi_\parallel = 0.273 \times \frac{48 \sqrt{\pi} e^3 k (2kT_e)^{3/2}}{e^3 \sqrt{m_e \ln \Lambda}} = 1.389 \frac{n_e ek}{m_e \tilde{v}_{ei}} \tag{5.1c}
\]

and

\[
\psi_\parallel = -3.203 \frac{kT_e}{e}, \text{ where } \tilde{v}_{ei} = n_i \frac{4 \sqrt{2\pi}}{3} \left( \frac{m_e}{kT_e} \right)^{3/2} \left( \frac{e^2}{4 \pi e^2} \frac{m_e}{e} \right)^2 \ln \Lambda. \tag{5.1d}
\]