properties are discussed briefly in Secs. 5 and 6. In Sec. 7 we consider various mixture rules that are useful for approximate calculations of electron transport properties and assess the accuracy of these formulas in comparison with the results of the rigorous theory of Sec. 4. Finally, in Sec. 8 we discuss the effects of nonelastic collisions.

2. WEAKLY IONIZED PLASMAS

When the degree of ionization of a plasma is sufficiently low, the electron-electron collision terms can be neglected in the electron Boltzmann equation and in the Cartesian-tensor equations of Chapter VII, Sec. 6. We refer to this situation as the Lorentzian limit. In this section we shall derive expressions for the electron transport properties that apply to such weakly ionized plasmas. As might be expected, omission of the electron-electron collision terms results in a considerable simplification in the calculation of transport properties. Furthermore, it is possible to obtain from a consideration of the Lorentzian limit the forms for the electron current density and heat flux that remain valid for arbitrary degrees of ionization, so that the discussion of this section serves as useful introduction to the general theory of transport properties. In fact, if electron-ion collisions are retained, the Lorentzian expressions give reasonable values for certain transport properties of partially and fully ionized plasmas. Thus, for example, for a fully ionized plasma the Lorentzian value of the electrical conductivity in the absence of a magnetic field differs by only a factor of approximately two from the exact value.

From equations (VII 6.8) it can be seen that the electron diffusion velocity, and hence current density, and the electron heat flux depend on the anisotropic part $f^1$ in the Cartesian-tensor expansion of the electron distribution function. Since $f^1$ is governed by the second of the Cartesian tensor equations, the “momentum” equation (VII 6.40), we consider first the solution of that equation, regarding for the present the isotropic part $f^0$ of the distribution function as unspecified.

The conditions under which the Lorentzian limit applies to equation (VII 6.40) can be estimated from equations (VII 6.35) and (VII 6.36) for the collision terms $C_{eh}^1$ and $C_{ee}^1$. From these equations it follows that with respect to order of magnitude we have

$$
\frac{|C_{ee}^1|}{|C_{eh}^1|} \sim \frac{n_e \Gamma_{ee}^0 f^1}{v_{eh} f^1 + v_{eh} U_{eh} \frac{\partial f^0}{\partial C}}.
$$

Here, since all terms in equation (VII 6.36) for $C_{ee}^1$ are of similar magnitude, we have based our estimate of $|C_{ee}^1|$ on the first term. We require only
a sufficient condition that the ratio (2.1) be small. Accordingly, we can for simplicity omit the second term in the denominator for the purposes of our estimate. The relative importance of this term will be discussed later in this section. Since the transport properties depend on electrons of thermal energies, that is, electrons for which \( \frac{1}{2}m_e C^2 \sim \frac{1}{2}kT_e \), it is appropriate to evaluate the ratio (2.1) in some average sense for thermal electrons. From the normalization condition

\[
1 = \int_{-\infty}^{\infty} f_e dC = 4\pi \int_0^{\infty} f^0 C^2 dC,
\]

we can write for thermal electrons

\[
f^0 \sim \frac{1}{C^3}.
\]

In terms of the electron-electron momentum-transfer collision frequency [see equation (VII 4.14)]

\[
\nu_{ee}^{(1)} = \frac{4n_e \Gamma_{ee}}{C^3} \sim \frac{4n_e \Gamma_{ee}}{C_e^3}
\]

the ratio (2.1) can then be expressed as

\[
\left| \frac{C_{ee}^1}{C_{eh}^1} \right| \sim \frac{\nu_{ee}^{(1)}}{\nu_{eh}} \sim \frac{\bar{\nu}_{ee}}{\bar{\nu}_{eh}}.
\]

Here, to be more definite, we have introduced the energy-weighted mean collision frequencies defined by the equation following (II 6.32). For \( \bar{\nu}_{eh} \), we have to lowest order in \( m_e/m_h \)

\[
\bar{\nu}_{eh} = \int_0^{\infty} \frac{4m_e}{kT_e} \nu_{eh}^{(1)}(C)f^0 4\pi C^2 dC. \tag{2.2a}
\]

With the use of equation (VII 4.10), \( \bar{\nu}_{ee} \) is given by

\[
\bar{\nu}_{ee} = n_e \frac{8\sqrt{\pi}}{3} \left( \frac{m_e}{kT_e} \right)^{3/2} \left( \frac{e^2}{4\pi e_0 m_e} \right)^2 \ln \Lambda \tag{2.2b}
\]

when \( f^0 \) is taken to be the Maxwellian distribution (3.3). It follows from the foregoing that the electron-electron collision term \( C_{ee}^1 \) can be neglected in equation (VII 6.40) for \( f^1 \) if

\[
\bar{\nu}_{ee} \ll \sum_h \bar{\nu}_{eh}. \tag{2.3}
\]
When the condition (2.3) is satisfied, equation (VII 6.40) becomes
\[
\frac{\partial}{\partial t} (n_e f^i) + CV(n_e f^0) + a' n_e \frac{\partial f^0}{\partial C} - \frac{e}{m_e} B \times n_e f^i = -n_e \sum_h \left[ v^{(1)}_{eh} e^2 \left( f^i + U_h \frac{\partial f^0}{\partial C} \right) \right].
\] (2.4)

To calculate the transport properties of a Lorentzian plasma, we must first solve equation (2.4) for \( f^i \) and then substitute the result in equations (VII 6.8) for the electron diffusion velocity and heat flux. [Since the contribution of the electrons to the viscous stress of the plasma as a whole is almost always small, we shall not consider the electron viscosity coefficient, which from equations (VII 6.8) depends on \( f^e_0 \).]

We first calculate the electrical conductivity of a uniform, steady-state plasma in the presence of constant electric and magnetic fields. Since for a uniform weakly ionized plasma, the diffusion velocity of the neutrals and the contribution of the ions to the sum on the right-hand side of equation (2.4) are negligible, we can omit the part of the collision term in equation (2.4) that involves \( U_h \). Under these conditions \( \alpha^e = -eE'/m_e \) [see equation (VII 6.15)], and equation (2.4) becomes
\[
\frac{e}{m_e} E' \frac{\partial f^0}{\partial C} + \frac{e}{m_e} B \times f^i = f^i \sum_h v^{(1)}_{eh}.
\] (2.5)

Equation (2.5) can be solved directly for \( f^i \) in terms of \( f^0 \). For this purpose we adopt the "natural" coordinate system defined by the three orthogonal vectors

\[
\begin{align*}
E_{\parallel} & \equiv (E \cdot b)b, \\
E'_{\perp} & \equiv E' - E_{\parallel},
\end{align*}
\]

and
\[
b \times E'
\] (2.6)

Here
\[
b \equiv B/B
\] (2.7)

is a unit vector the direction of the magnetic field. The vector \( f^i \) can then be written in terms of its components in these directions as
\[
f^i = [f^i_{\parallel} E_{\parallel} + f^i_{\perp} E'_{\perp} + f^i_{H}(b \times E')] \left( e/m_e \right),
\] (2.8)

where the factor \( e/m_e \) is introduced for later convenience. The subscripts \( \parallel, \perp, \) and \( H \) denote the directions parallel to and perpendicular to the
magnetic field, and the “Hall” direction, which is perpendicular to both
B and E'. The coefficients $f_\parallel^1, f_\perp^1, f_H^1$ can be found from the substitution
of $f^1$ in the form (2.8) into equation (2.5). This yields, after division by $e/m_e$,

$$(E_\parallel + E_\perp) \frac{\partial f^0}{\partial C} + (\omega_e f_\parallel^1 b \times E' - \omega_e f_H^1 E'_\parallel)$$

$$= v_{eh}(f_\parallel^1 E_\parallel + f_\perp^1 E_\perp + f_H^1 b \times E')$$

where the electron cyclotron frequency $\omega_e$ is

$$\omega_e = eB/m_e \quad (2.9)$$

and where

$$v_{eh} \equiv \sum_\nu v_{eh}^{(1)} \quad (2.10)$$

Equating coefficients of the three orthogonal vectors, we obtain

$$f_\parallel^1 = \frac{1}{v_{eh}} \frac{\partial f^0}{\partial C},$$

$$v_{eh} f_\parallel^1 + \omega_e f_\parallel^1 = \frac{\partial f^0}{\partial C}, \quad (2.11a)$$

and

$$v_{eh} f_H^1 = \omega_e f_H^1.$$  

From the last two of these relations, $f_\parallel^1$ and $f_H^1$ are given by

$$f_\parallel^1 = \frac{v_{eh}}{v_{eh}^2 + \omega_e^2} \frac{\partial f^0}{\partial C} \quad (2.11b)$$

and

$$f_H^1 = \frac{\omega_e}{v_{eh}^2 + \omega_e^2} \frac{\partial f^0}{\partial C}. \quad (2.11c)$$

With these results and equation (VII 6.8) for the electron diffusion
velocity $U_e$, we can now calculate the contribution $J_e$ of the electrons to the
current-density vector J. Thus we have

$$J_e \equiv -en_e U_e = -en_e \frac{4\pi}{3} \int_0^\infty C^4 \mu^1 dC. \quad (2.12)$$

As indicated in Chapter IV, Sec. 3, except for the case of ion slip in very
strong magnetic fields, the ion current density for a uniform plasma is
much less than the electron current density, so that $J \approx J_e$. (The ion
current density can also be important in the region of nonuniform plasma in the vicinity of a solid surface; see Chapter III, Secs. 6 and 7.) Over a wide range of conditions, then, to calculate the overall electrical conductivity of a weakly ionized plasma we need consider only the foregoing solution of the electron Boltzmann equation, which is mathematically decoupled from the Boltzmann equations for the other species. The result from the substitution of equations (2.8) and (2.11) in equation (2.12) for \( \mathbf{J}_e \) can be expressed in terms of the following Ohm's-law relation [cf. equation (IV 3.21a)]:

\[
\mathbf{J}_e = \sigma_{\|} \mathbf{E}_{\|} + \sigma_{\perp} \mathbf{E}_{\perp} + \sigma_H \mathbf{b} \times \mathbf{E}.
\]  

(2.13)

The components \( \sigma_{\|} \), \( \sigma_{\perp} \), and \( \sigma_H \) of the so-called tensor conductivity are then

\[
\sigma_{\|} = \frac{4 \pi n_e e^2}{3 m_e} \int_0^\infty \frac{C^3}{v_{eh}^2 + \omega_e^2} \left( -\frac{\partial f_0}{\partial C} \right) dC,
\]

(2.14a)

\[
\sigma_{\perp} = \frac{4 \pi n_e e^2}{3 m_e} \int_0^\infty \frac{C^3 v_{eh}}{v_{eh}^2 + \omega_e^2} \left( -\frac{\partial f_0}{\partial C} \right) dC,
\]

(2.14b)

and

\[
\sigma_H = \frac{4 \pi n_e e^2}{3 m_e} \int_0^\infty \frac{C^3 \omega_e}{v_{eh}^2 + \omega_e^2} \left( -\frac{\partial f_0}{\partial C} \right) dC.
\]

(2.14c)

In the absence of a magnetic field, \( \omega_e = 0 \) and the perpendicular and parallel conductivities \( \sigma_{\perp} \) and \( \sigma_{\|} \) are equal. In this case the current density is given by

\[
\mathbf{J}_e = \sigma_{\|} \mathbf{E}.
\]

Thus, the conventional scalar conductivity is identical to \( \sigma_{\|} \) and, in the Lorentzian limit, can be found from equation (2.14a).

The evaluation of expressions (2.14) for the Lorentzian conductivity requires a knowledge of the isotropic part \( f^0 \) of the distribution function. Since from equations (2.11) \( f^1 \) is now known in terms of \( f^0 \), the function \( f^0 \) can be obtained from the first of the Cartesian-tensor equations, equation (VII 6.39). Under the present conditions for a weakly ionized plasma, this equation becomes

\[
-\frac{e}{3m_e} \mathbf{E} \cdot \frac{d}{dC} \left( C^2 f^1 \right) = \sum_{\alpha=1}^3 \frac{m_e}{k_B m_\alpha} \frac{d}{dC} \left[ C^3 \left( f^0 + \frac{kT_\alpha}{m_\alpha C} \frac{1}{m_e} \frac{df^0}{dC} \right) \right].
\]

(2.15)

Here we have omitted electron-electron collision term \( C_{ee}^1 \) in comparison with the electron/heavy-particle collision terms \( C_{eh}^1 \). By means of an estimate similar to that used to obtain the relative magnitudes (2.2) of \( C_{ee}^1 \) and
Section 2  Weakly Ionized Plasmas 375

C.e [see equation (3.6) in the following section], the condition that C.e be neglected in equation (2.15) is

\[ \bar{v}_{ee} \ll \sum_k \frac{m_e}{m_h} \bar{v}_{eh}. \]  

(2.16)

Clearly this condition is much more restrictive than the condition (2.3) for the validity of the foregoing Lorentzian analysis for f^1.

To solve equation (2.15) for f^0 we must first evaluate E' \cdot f^1. From equations (2.8) and (2.11) we have

\[ E' \cdot f^1 = \frac{e}{m_e} (f_{1//} E'_{//} + f_{1//} E'_{//}) \]

\[ = \frac{e}{m_e} \left( E_{//}^2 + \frac{v_{eh}^2}{v_{eh}^2 + \omega_z^2} E_{//}^2 \right) \frac{1}{v_{eh}} \frac{df^0}{dC}. \]

With the use of this result, integration of equation (2.15) with respect to C gives

\[ \frac{df^0}{dC} = \sum_h \frac{(m_e/m_h) v_{eh}^4 C f^0}{(kT_h/m_h) v_{eh}^4 + \frac{1}{3} \left( e^2/m_e^2 \right) \left( E_{//}^2 + E_{//}^2 \frac{v_{eh}^2}{v_{eh}^2 + \omega_z^2} \right)^1/v_{eh}}, \]

(2.17)

where the constant of integration has been set equal to zero on the basis of the condition that both f^0 and df^0/dC vanish as C \to \infty. Equation (2.17) can be formally integrated to yield, for a value of T_h that is the same for all heavy species,

\[ f^0(C) = f^0(0) \exp \left\{ -\int_0^C \frac{m_e C \ dC}{kT_h + \frac{1}{3} \left( e^2/m_e^2 \right) \left( E_{//}^2 + E_{//}^2 \frac{v_{eh}^2}{v_{eh}^2 + \omega_z^2} \right)^1/v_{eh}} \right\}. \]

(2.18)

The integration constant f^0(0) can be found from the normalization condition

\[ \int_{-\infty}^{\infty} f^0(C) \ d^3C = 4\pi \int_0^\infty f^0(C) C^2 \ dC = 1. \]

[From the first of equations (VII 6.19), terms other than f^0 in the Cartesian-tensor expansion do not contribute to this integral.]

In equation (2.18) for f^0, when the field E' is sufficiently small the second term in the denominator of the integral can be neglected. In this case f^0 reduces to a Maxwellian function with the electron temperature equal to the heavy-particle temperature. For strong fields, the electron temperature is greater than T_h and f^0 is not, in general, Maxwellian.
To evaluate explicitly the Lorentzian distribution (2.18) and the conductivity expressions (2.14), particular forms must be taken for $\nu_{eh}^{(1)}$ (see, for example, Chapman and Cowling, 1952, Secs. 18.71 and 18.73).

We now return to equation (2.4) for $f^1$ and consider separately the effects on the solution of that equation of heavy-particle diffusion velocities, high-frequency electric fields, and spatial gradients.

For a uniform, steady-state plasma retention of the part of the collision term in equation (2.4) that involves the heavy-particle diffusion velocities $U_h$ results in the addition of a term

$$\sum_h \nu_{eh}^{(1)} U_h \frac{\partial f^0}{\partial C}$$

(2.19)

to equation (2.5). Since this equation is linear in $f^1$, the solution for $f^1$ can be written as the sum of the previous solution [(2.8) and (2.11)], which depends on the electric field, and the solution of the equation

$$- \sum_h \nu_{eh}^{(1)} U_h \frac{\partial f^0}{\partial C} + \frac{e}{m_e} B \times f^1 = f^1 \sum_h \nu_{eh}^{(1)}.$$

By means of an analysis similar to that used to obtain equations (2.8) and (2.11), this solution is

$$f^1 = - \sum_h \nu_{eh}^{(1)} (f^1_{\parallel}(U_h)_{\parallel} + f^1_{\perp}(U_h)_{\perp} + f^1_{\times} b \times U_h),$$

(2.20)

where

$$(U_h)_{\parallel} \equiv (U_h \cdot b)b$$

and

$$(U_h)_{\perp} \equiv U_h - (U_h)_{\parallel},$$

and where $f^1_{\parallel}$, $f^1_{\perp}$, and $f^1_{\times}$ are again given by equations (2.11). From this result and equation (2.12) for the electron diffusion velocity $U_e$, we can estimate the contribution to $U_e$ that results from the retention of $U_h$ in the collision term in equation (2.4). For a particular heavy-particle species this contribution is of the order of (see Exercise 2.2)

$$(U_e)_{\parallel} \approx \frac{\nu_{eh}^{(1)}}{v_{eh}} (U_h)_{\parallel},$$

$$(U_e)_{\perp} \approx \frac{1}{1 + \beta_e^2 \nu_{eh}^{(1)}} (U_h)_{\perp},$$

(2.21)

and

$$(U_e)_{\times} \approx \frac{\beta_e^2 \nu_{eh}^{(1)} b \times U_h}{1 + \beta_e^2 \nu_{eh}^{(1)}}.$$
Section 2 weakest ionized plasmas 377

where

\[ \beta_e \sim \frac{\omega_e}{\nu_{ei}} \].

To the extent that the heavy-particle diffusion velocities are small, this effect can then be neglected. Accordingly, we shall omit the parts of \( C_{eh} \) that involves \( U_h \) in our further calculations.

We now consider the technically more important problem of the response of a uniform, weakly ionized plasma to a high-frequency electric field (see Chapter III, Sec. 8). In this analysis, we include the effect of a constant magnetic field. If it is assumed that high-frequency time variation of the electron number density can be neglected, the pertinent form of equation (2.4) for \( \mathbf{f}^i \) is

\[
\frac{\partial \mathbf{f}^i}{\partial t} - \frac{e}{m_e} \mathbf{E}' \frac{\partial \rho^0}{\partial C} - \frac{e}{m_e} \mathbf{B} \times \mathbf{f}^i = -\mathbf{f}^i \sum_{h} \nu_{eh}^{(1)}. \tag{2.22}
\]

We now represent the electric field by

\[ \mathbf{E}' = \mathbf{E}^0 + \mathbf{E}^1 \operatorname{Re} e^{-i\omega t}, \]

where \( i^2 = -1 \), \( \mathbf{E}^0 \) is a constant (d.c) field, and \( \mathbf{E}^1 \) is the (real) amplitude of the component of the field of frequency \( \omega \). In accord with our previous solution (2.18) for \( f^0 \), if \( \mathbf{E}^1 \) is sufficiently small we can neglect the time variation of \( f^0 \) in equation (2.22). Our previous solution (2.8) and (2.11) then applies to the part of \( \mathbf{f}^i \) associated with the constant field \( \mathbf{E}^0 \), and the corresponding current density is given by equations (2.13) and (2.14), as before. The additional effect of the high-frequency field is obtained from the solution of equation (2.22) with \( \mathbf{E}' \) replaced by

\[ \mathbf{E}' = \mathbf{E}^1 \operatorname{Re} e^{-i\omega t}. \tag{2.23} \]

To effect this solution we write for the high-frequency part of \( \mathbf{f}^i \)

\[
\mathbf{f}^i = \operatorname{Re} \left\{ (\tilde{\mathbf{f}}_{\parallel} \mathbf{E}_{\parallel}^0 + \tilde{\mathbf{f}}_{\perp} \mathbf{E}_{\perp}^0 + \tilde{\mathbf{f}}_{\times} \mathbf{b} \times \mathbf{E}^1) \frac{e}{m_e} e^{-i\omega t} \right\}. \tag{2.24}
\]

where \( \mathbf{E}_{\parallel}^0 = (\mathbf{E}^0 \cdot \mathbf{b}) \mathbf{b} \) and \( \mathbf{E}_{\perp}^0 = \mathbf{E}^0 - \mathbf{E}_{\parallel}^0 \). The complex coefficients \( \tilde{\mathbf{f}}_{\parallel} \), \( \tilde{\mathbf{f}}_{\perp} \), and \( \tilde{\mathbf{f}}_{\times} \) can be found from the substitution of equations (2.23) and (2.24) in equation (2.22). Equating coefficients in the directions of the three orthogonal vectors \( \mathbf{E}_{\parallel}^0 \), \( \mathbf{E}_{\perp}^0 \), and \( \mathbf{b} \times \mathbf{E}^1 \), we thus obtain

\[
(-i\omega + \nu_{ei} \tilde{f}_{\parallel}) \tilde{f}_{\parallel}^0 = \frac{\partial \rho^0}{\partial C},
\]

\[
(-i\omega + \nu_{ei} \tilde{f}_{\perp}^0 + \omega_s \tilde{f}_{\parallel}^0 = \frac{\partial \rho^0}{\partial C},
\]
and

\((-i\omega + v_{eh})f^1_H = \omega_e f^1_H.\)

Again we have used the total collision frequency \(v_{eh} \equiv \sum v_{e1}\). The solution of preceding equations for the three components of \(f^1\) can be expressed as follows:

\[f^1_H = \frac{v_{eH} + i\omega}{v_{eH} + \omega^2} \frac{\partial f^0}{\partial C}.\]

\[f^1_H = \frac{\omega_e}{(v_{eH} - i\omega)^2 + \omega_e^2} \frac{\partial f^0}{\partial C},\]

\[= -i \left[ \frac{1}{v_{eH} - i(\omega + \omega_e)} - \frac{1}{v_{eH} - i(\omega - \omega_e)} \right] \frac{\partial f^0}{\partial C},\]

and

\[f^1_\perp = \frac{\omega_e}{(v_{eH} + \omega_e)^2 + \omega_e^2} \frac{\partial f^0}{\partial C},\]

Here the final forms for \(f^1_H\) and \(f^1_\perp\) have been written with our previous equations (2.11) in mind, for reasons that will become apparent shortly.

With the use of equation (2.12) for \(J_e\), the current density associated with the high-frequency field can be expressed in terms of complex conductivities \(\sigma_{\parallel}, \sigma_{\perp},\) and \(\sigma_H\) as

\[J_e = \text{Re}[\sigma_{\parallel} E_\parallel + \sigma_{\perp} E_\perp + \sigma_H b \times E_\parallel e^{-i\omega t}].\] (2.26)

Thus, for example, the real and imaginary parts of \(\sigma_{\parallel}\) determine respectively the in-phase and out-phase components of \(J_e^\parallel\). In general, the total current density is the sum of the current densities given by equations (2.26) and (2.13).

Comparison of equations (2.25) and (2.11) shows that the present complex conductivities can be written in terms of our previous dc conductivities (2.14) in the forms

\[\sigma_{\parallel} = \sigma_{\parallel} (\omega) + i\sigma_H (\omega),\]

\[\sigma_{\perp} = \frac{1}{2} \left\{ \sigma_{\perp} (\omega + \omega_e) + \sigma_{\perp} (\omega - \omega_e) + i[\sigma_H (\omega + \omega_e) + \sigma_H (\omega - \omega_e)] \right\},\] (2.27)
and
\[ \tilde{\sigma}_H = \frac{1}{2} [\sigma_H(\omega + \omega_e) - \sigma_H(\omega - \omega_e) - i[\sigma_\perp(\omega + \omega_e) - \sigma_\perp(\omega - \omega_e)]] \].

Here, for example, \( \sigma_\perp(\omega + \omega_e) \) represents the value of \( \sigma_\perp \) as given by equation (2.14b) with \( \omega_e \) in that equation replaced by \( \omega + \omega_e \). Similar interpretations apply to the other terms in equations (2.27). It follows that to obtain the current density as a result of both constant and high-frequency electric fields and in the presence of a constant magnetic field, it is necessary only to calculate \( \sigma_\perp \) and \( \sigma_H \) as functions of \( \omega_e \) from equations (2.14b) and (2.14c). (As we have already seen, \( \sigma_\parallel \) is equal to the value of \( \sigma_\parallel \) when \( \omega_e = 0 \).) It turns out, in fact, that for partially and fully ionized plasmas, where the effects of electron-electron collisions must be included, the present result in the form of equations (2.27) still applies, although the values of \( \sigma_\perp \) and \( \sigma_H \) are modified from those given by equations (2.14). With regard to the propagation of electromagnetic waves, equation (2.2b) in the \( B = 0 \) limit
\[ J_e = \text{Re} \tilde{\sigma}_\parallel E_\parallel e^{-i\omega t} \]
can be used in place of equation (III 8.4b) to obtain a more accurate form of the dispersion relation (III 8.5a).

We now consider the effects of spatial gradients. As before, we analyze these effects separately, so that the appropriate form of equation (2.4) for \( \mathbf{f}^\parallel \) is
\[ -C \nabla (n_e f^0) + \frac{e}{m_e} E \times n_e \mathbf{f}^\parallel = n_e \nu_{eh} \mathbf{f}^1. \]  
(2.28)

By analogy with our previous results, as given for example by equations (2.8) and (2.11), the solution to equation (2.28) is
\[ \mathbf{f}^1 = f^1_\parallel \nabla \ln (n_e f^0) + f^1_\perp \nabla \ln (n_e f^0) + f^1_h \mathbf{b} \times \nabla \ln (n_e f^0), \]  
(2.29)
where
\[ f^1_\parallel = -\frac{1}{\nu_{eh}} C f^0, \]
\[ f^1_\perp = -\frac{\nu_{eh}}{\nu_{eh}^2 + \omega_e^2} C f^0, \]  
(2.30)
and
\[ f^1_h = -\frac{\omega_e}{\nu_{eh}^2 + \omega_e^2} C f^0. \]
In equation (2.28) the as-yet-unspecified vector
\[ \nabla \ln (n_e f^0) = \frac{\partial}{\partial x} \ln (n_e f^0) \]
is regarded as the "driving force" which is analogous to the electric field in our previous calculations. The components of this vector that are parallel to and perpendicular to the magnetic field are denoted in equation (2.29) by \( \nabla_{\parallel} \) and \( \nabla_{\perp} \), respectively.

From the solution (2.29) and (2.30) we can now calculate the electron heat flux which results from the spatial gradient \( \nabla \ln (n_e f^0) \). The heat flux vector is given in terms of \( f^1 \) by equations (VII 6.8) as
\[ q_e = \frac{2\pi}{3} n_e m_e \int_0^\infty C^3 f^1 \, dC \]  
(2.31)

Although it would not be difficult to express \( q_e \) in terms of \( \nabla \ln (n_e f^0) \), we shall consider only the particular case where \( f^0 \) is a Maxwellian function of the electron temperature \( T_e \). By direct differentiation of \( f_M \) from equation (VII 2.2) we can then write
\[ \nabla \ln (n_e f^0) = \nabla \ln (n_e f_M) = \nabla \ln p_e - \nabla \ln T_e + \nabla \ln f_M \]  
(2.32)

Here we have introduced the electron pressure \( p_e = n_e k T_e \) so that our results will be expressed in a conventional form.

We now substitute \( f^1 \) from equations (2.29), (2.30), and (2.32) in equation (2.31) to obtain an expression for \( q_e \). For the moment, we shall consider only the effect of temperature gradients and omit the term \( \nabla \ln p_e \) in equation (2.32). The part in question of electron heat flux can then be written in the form
\[ q_e = -\lambda_{\parallel} \nabla_{\parallel} T_e - \lambda_{\perp} \nabla_{\perp} T_e - \lambda_H b \times \nabla T_e, \]  
(2.33)

where the electron thermal-conductivity coefficients are given by
\[ \lambda_{\parallel} = \frac{2\pi}{3} n_e m_e \int_0^\infty \frac{1}{\nu_{elH}} \left( \frac{m_e C^2}{2k T_e} - \frac{5}{2} \right) C^6 f_M \, dC, \]
\[ \lambda_{\perp} = \frac{2\pi}{3} n_e m_e \int_0^\infty \frac{\nu_{elH}}{\nu_{elH}^2 + \omega_e^2} \left( \frac{m_e C^2}{2k T_e} - \frac{5}{2} \right) C^6 f_M \, dC, \]  
(2.34)
and

$$\lambda_H = \frac{2\pi n_e m_e}{3 T_e} \int_0^\infty \frac{\omega_e^2}{\nu_{eH}^2 + \omega_e^4} \left( m_e C^2 \frac{5}{2kT_e} \right) C^6 fM dC.$$  

From these results it can be seen that, as in the case of electrical conductivity, a magnetic field acts to reduce the perpendicular thermal conductivity.

In the foregoing we have considered separately the diffusion, and hence current flow, caused by an electric field and the heat flux caused by a temperature gradient. Since equation (2.4) is linear in $f^l$, we have noted that the consequences of the various driving forces which we have considered are additive. Beyond this, since by equations (2.12) and (2.31) both $J_e$ and $q_e$ depend on the same function $f^l$, there are additional effects of these driving forces. That is, by equation (2.31), the part (2.8) of $f^l$ that results from an electric field will contribute also to the heat flux and, by equation (2.12), the part (2.29) of $f^l$ that results from a temperature gradient will contribute also to diffusion. This phenomenon, which is difficult to understand on the basis of simple mean-free-path arguments, is known as thermal diffusion.

It is a straightforward matter to generalize our previous analysis to account for the simultaneous action of fields and spatial gradients and to account for thermal diffusion. In these calculations we shall assume that $f^0$ is a Maxwellian function (VII 2.2) of $T_e$, so that in equations (2.11) and (2.14) we have, by differentiation, $-\partial f^0/\partial C = m_e C fM/kT_e$. This condition makes our present results more directly useful for our discussion in Sec. 4 of partially ionized plasmas.

If we do not include here the effect of high-frequency electric fields [and the effect of the part of the collision term in equation (2.4) that involves $U_i$] the appropriate solution to equation (2.4) can be written in the form (see Exercise 2.3)

$$f^l = C fM \left[ A_1 V_\parallel \ln T_e + A_2 V_\perp \ln T_e + A_3 b \times V \ln T_e \right]$$

$$- \frac{e}{kT_e} \left[ D_1 \mathcal{E}_\parallel + D_2 \mathcal{E}_\perp + D_3 b \times \mathcal{E} \right].$$

(2.35)

Here the "generalized electric field"

$$\mathcal{E} = E' + \frac{\nabla p_e}{en_e} + \frac{m_e}{e} \frac{Du}{Dt}$$

$$\simeq E' + \frac{\nabla p_e}{en_e}$$

(2.36)
incorporates the term $\nabla p_e$ from equation (2.32) and the term $D_{\mathbf{u}}/Dt$ in the acceleration $a'$ in equation (2.4) [see equation (VII 6.15)]. The indicated approximation results from the fact that in almost all situations the last term in $\mathcal{E}$ is negligible.

The coefficients $A_1, A_2, A_3,$ and $D_1, D_2, D_3$ in equation (2.35) are given by [cf. equations (2.11) and (2.30)]

$$A_1 = -\frac{1}{v_{eH}} \left( \frac{m_e C^2}{2kT_e} - \frac{5}{2} \right),$$

$$A_2 = -\frac{v_{eH}}{v_{eH}^2 + \omega_e^2} \left( \frac{m_e C^2}{2kT_e} - \frac{5}{2} \right),$$

$$A_3 = -\frac{\omega_e}{v_{eH}^2 + \omega_e^2} \left( \frac{m_e C^2}{2kT_e} - \frac{5}{2} \right).$$

$$D_1 = \frac{1}{v_{eH}},$$

$$D_2 = \frac{v_{eH}}{v_{eH}^2 + \omega_e^2},$$

and

$$D_3 = \frac{\omega_e}{v_{eH}^2 + \omega_e^2}.$$

Substitution of equations (2.35) and (2.37) for $\mathbf{f}^1$ in equations (2.31) and (2.12) for $q_e$ and $J_e$ then yields the following expressions for the electron heat-flux and current-density vectors:

$$q_e = -\lambda_\parallel \mathbf{V}_\parallel T_e - \lambda_\perp \mathbf{V}_\perp T_e - \hat{\lambda}_H \mathbf{b} \times \nabla T_e$$

$$- T_e \left( \phi_\parallel + \frac{5k}{2e} \sigma_\parallel \right) \mathcal{E}_\parallel - T_e \left( \phi_\perp + \frac{5k}{2e} \sigma_\perp \right) \mathcal{E}_\perp$$

$$- T_e \left( \phi_H + \frac{5k}{2e} \sigma_H \right) \mathbf{b} \times \mathcal{E}$$

(2.38)

and

$$J_e = \sigma_\parallel \mathcal{E}_\parallel + \sigma_\perp \mathcal{E}_\perp + \sigma_H \mathbf{b} \times \mathcal{E} + \phi_\parallel \mathbf{V}_\parallel T_e + \phi_\perp \mathbf{V}_\perp T_e + \phi_H \mathbf{b} \times \nabla T_e.$$

In equations (2.38) the electron thermal and electrical conductivities $\lambda$ and $\sigma$ are given by equations (2.34) and (2.14) as before, with the exception
that in equations (2.14) $-\partial f^0/\partial C$ is now to be replaced by $m_e C f_M/k T_e$. The thermal-diffusion coefficients $\phi_\parallel$, $\phi_\perp$, and $\phi_H$ are

$$\phi_\parallel = \frac{4\pi e n_e}{3 T_e} \int_0^\infty \frac{1}{v_{eh}} \left( \frac{m_e C^2}{2k T_e} - \frac{5}{2} \right) C f_M dC,$$

$$\phi_\perp = \frac{4\pi e n_e}{3 T_e} \int_0^\infty \frac{v_{eh}}{v_{eh}^2 + \omega_e^2} \left( \frac{m_e C^2}{2k T_e} - \frac{5}{2} \right) C f_M dC,$$

and

$$\phi_H = \frac{4\pi e n_e}{3 T_e} \int_0^\infty \frac{\omega_e}{v_{eh}^2 + \omega_e^2} \left( \frac{m_e C^2}{2k T_e} - \frac{5}{2} \right) C f_M dC. \quad (2.39)$$

From equations (2.39) it is easily shown that the thermal-diffusion coefficients vanish if $v_{eh}$ is a constant, independent of $C$. If $v_{eh}$ increases monotonically with $C$, $\phi_\parallel$ and $\phi_H$ are negative, whereas if $v_{eh}$ decreases monotonically with $C$, $\phi_\parallel$ and $\phi_H$ are positive. If $v_{eh}$ varies monotonically with $C$, the behavior of $\phi_\parallel$ for weak magnetic fields is the same as that of $\phi_H$. For sufficiently strong magnetic fields, however, the sign of $\phi_\perp$ will be reversed.

One might suppose that as a result of thermal diffusion two new sets of coefficients would be required, one relating $q_e$ to $\delta$ and the other relating $J_e$ to $VT_e$. Direct evaluation of $q_e$ and $J_e$ from equations (2.35) and (2.37) shows, however, that a single set (2.39) of thermal-diffusion coefficients is sufficient. We note at this point that our use of the symbol $\tilde{\lambda}$ rather than $\lambda$ for thermal conductivity is to distinguish our present form for $q_e$ in terms of $VT_e$ and $\delta$ from the more usual representation of $q_e$ in terms of $VT_e$ and $J_e$. This distinction will be discussed in greater detail in Sec. 4.

We have chosen to write $q_e$ and $J_e$ in terms of vector equations with scalar coefficients, such as $\sigma_\parallel$, $\sigma_\perp$, and $\sigma_H$. An alternate approach is to introduce tensor coefficients so that $q_e$ and $J_e$ can be expressed more compactly as

$$q_{e\beta} = -\tilde{\lambda}_{e\beta} \frac{\partial T_e}{\partial x_\beta} - T_e \left( \phi_{e\beta} + \frac{5 k}{2} \sigma_{e\beta} \right) \delta^{\beta\gamma}$$

and

$$J_{e\beta} = \sigma_{e\beta} \delta^{\beta\gamma} + \phi_{e\beta} \frac{\partial T_e}{\partial x_\beta}. \quad (2.40)$$

If we adopt the coordinate system shown in Fig. 1, in which the $x_3$ direction is parallel to $b$ and the orientation of the vector $\delta$ is arbitrary, $\sigma_{e\beta}$ can be expressed in terms of $\sigma_\parallel$, $\sigma_\perp$, and $\sigma_H$ by means of the matrix

$$\{\sigma_{e\beta}\} = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}. \quad (2.41)$$
Here it is to be understood that the first subscript, \( a \), denotes the row of the matrix and the second subscript, \( \beta \), the column. Similar relations hold for \( \lambda_{ab} \) and \( \phi_{ab} \) (see Exercise 2.4).

As we indicated earlier, the results of this section are useful not only as they apply directly to weakly ionized plasmas, but also in that the present description remains qualitatively valid for partially ionized plasmas as well. Before discussing the theory for partially ionized plasmas, we first establish the conditions for which \( f^0 \) is Maxwellian.

**Exercise 2.1.** Obtain simplified expressions for the various transport coefficients (2.14), (2.34), and (2.39) that apply when \( \nu_{ab}^{(1)} \) is taken to be independent of \( C \). Compare your results with the mean-free-path values of \( \sigma \) given by equations (II 13.7b and IV 3.11).

**Exercise 2.2.**
1. Obtain the solution (2.20) for the effect of the heavy-particle diffusion velocities on \( f^1 \).
2. Use this result to express the corresponding contributions to \( U_s \) in terms of integrals similar in form to equations (2.14).
3. Simplify the results of part 2 for the case where each of the collision frequencies \( \nu_{ab}^{(1)} \) is taken to be independent of \( C \). With the use of integration by parts, you should recover equations (2.21).

**Exercise 2.3.** Provide the missing steps in the derivation of equations (2.35) through (2.39) for \( q_e, J_e, \) and the various transport properties. Show that it is necessary to introduce only one set of thermal-diffusion coefficients, that given by equations (2.39).
Exercise 2.4. Derive expressions similar to equation (2.41) for the tensor thermal-conductivity and thermal-diffusion coefficients $\lambda_{\lambda\phi}$ and $\phi_{\lambda\phi}$ that appear in equations (2.40).

3. THE CONDITIONS FOR A MAXWELLIAN DISTRIBUTION

In this section we consider the conditions under which the electron-electron collision term $C_{ee}^0$ is the dominant term in the first Cartesian-tensor equation (VII 6.39), the speed-dependent energy equation

$$\frac{D}{Dt} (n_e f^0) + \frac{C}{3} \nabla (n_e f^0) - \frac{e n_e}{3 m_e C^2} \mathbf{E}' \cdot \frac{\partial}{\partial C} (C^2 f^0) - \frac{C}{3} \nabla \cdot \mathbf{w}_e \frac{\partial f^0}{\partial C} = \sum_h C_{eh}^0 + C_{ee}^0. \quad (3.1)$$

Under such conditions this equation, with $C_{ee}^0$ given by equation (VII 6.32), is satisfied by

$$C_{ee}^0 = n_e^2 \Gamma_{ee} \frac{1}{C^2} \frac{\partial}{\partial C} \left[ f^0 f^0_0 + C \frac{\partial f^0}{\partial C} (J^0_2 + J^0_{-1}) \right] = 0, \quad (3.2)$$

where the integrals $I^0_2$ and $J^0_2$ are defined by equations (VII 6.33). That is, if all other terms in equation (3.1) are small when compared with the individual parts of $C_{ee}^0$, such as $(n_e^2 \Gamma_{ee}/C^2) \partial(f^0 f^0_0)/\partial C$, then equation (3.1) can only be satisfied if $C_{ee}^0$ is itself zero.

The solution to equation (3.2), as can be verified by direct substitution, is that $f^0$ is given by the Maxwellian distribution

$$f^0 = f_M = \left( \frac{m_e}{2 \pi k T_e} \right)^{3/2} e^{-m_e C^2/2 k T_e}. \quad (3.3)$$

Here the electron temperature $T_e$ is to be regarded as determined by the (speed-integrated) electron energy equation, which we shall discuss later in this section.

To assess the conditions of applicability of the Maxwellian distribution (3.3) we first ask when $C_{ee}^0$ will dominate the other collision terms in equation (3.1). Since we shall apply the distribution (3.3) to the calculation of transport properties, we consider the magnitude of the collision terms for electrons of thermal energies, as in the previous section. From equations (3.2) and (VII 6.33), the order of magnitude of each of the terms in $C_{ee}^0$ is

$$C_{ee}^0 \sim n_e^2 \Gamma_{ee}(f^0)^2 \sim n_e v_{ee}^2 f^0. \quad (3.4)$$