this result in the following section, several terms in equation (5.17) make no 
contribution to our calculations, so that the application of $C_{eh}$ is less 
complicated than it would appear from equation (5.17). In particular the 
derivatives of $v_{eh}$ in the first term do not contribute to the expressions 
that we shall use in the calculation of transport properties.

Exercise 5.1. Equation (5.4) gives the vector $V$ in terms of $C$ and $k$. 
Write an expression for $C$ in terms of $V$ and $k$.

Exercise 5.2. If $f_e(C)$ is an isotropic function $f_e(C)$ of electron speed, 
what is the value of $C_{eh}$ as given by the zero-order approximation (5.9)?

Exercise 5.3. Find the simplified form for the collision integral $C_{eh}$ from 
equation (5.16) that applies when the derivatives of $v_{eh}$ and the heavy-particle 
viscous stress can be neglected.

Exercise 5.4. Demonstrate the validity of the equalities

$$2 \int_{\Omega_k} k(g \cdot K) g I(g, x) \, d\Omega = g g Q^{(1)}_{ab}(g)$$

and

$$2 \int_{\Omega_k} k(C \cdot k) C \tilde{I}(C, C \cdot k) \, d\Omega_k = C C Q^{(1)}_{ab}(C),$$

which will be used in the next section. [The momentum-transfer cross 
section $Q^{(1)}_{ab}$ is defined by equation (4.9).]

6. THE CARTESIAN-TENSOR EXPANSION

In the preceding two sections we have discussed simplifications of the 
collision integrals that result from the general physical effects of small 
deflections in charged-particle collisions and the small electron mass. We 
now consider a particular representation of the electron Boltzmann equation 
which applies to a wide range of physical situations in collision-dominated 
plasmas. The formulation of this section will be applied in the following 
chapter to the discussion of the transport properties of a partially ionized 
plasma. The electron Boltzmann equation, which is amenable to particular 
simplifications, plays an important and distinct role in the study of these 
phenomena.

In a collision-dominated plasma, the electron distribution function $f_e(C)$
characteristically differs by only a small amount from an isotropic function 
$f^0(C)$ which is dependent on the electron velocity $C$ only through the 
speed $C$. This fact motivates the expansion of $f_e(C)$ with respect to its 
dependence on $C$ for small departures from an isotropic distribution. The
situation is somewhat similar to that with a collision-dominated mixture of neutral gases, where the isotropic functions are then equilibrium Maxwellian distributions. In a plasma, however, as a result of the relatively weak coupling between the electrons and the heavy particles the isotropic function $f^0(C)$ is not necessarily the equilibrium distribution.

The analysis of this section could be expressed in terms of either of two equivalent expansions for the angular dependence in velocity space of $f_e(C)$. The spherical-harmonic expansion, which has been used by Allis (1956) and others (see Shkarofsky, et al., 1966, Sec. 3.1), can be written as

$$f_e(C) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(C) Y_{lm}(\theta, \phi), \quad \text{(6.1)}$$

where $C$, $\theta$, and $\phi$ are spherical polar coordinates in velocity space (Fig. 4) and $Y_{lm}(\theta, \phi)$ are the spherical harmonic functions (see, e.g., Jackson, 1962, Sec. 3.4). The expansion coefficients $f_{lm}$ are in general complex functions of the electron speed $C$ and of position and time. Since $f_e(C)$ is a real function, the real part of the expansion (6.1) is implied and the coefficients $f_{l,-m}$ are related to the complex conjugates $f_{l,m}^*$ by

$$f_{l,-m} = (-1)^m f_{l,m}^*. \quad \text{(6.2)}$$

The first few terms in the expansion (6.1) are

$$f_e(C) = \frac{1}{\sqrt{4\pi}} f_{00}(C) - \sqrt{\frac{3}{8\pi}} \text{Re}[f_{11}(C)] \sin \theta \cos \phi$$

$$+ \sqrt{\frac{3}{8\pi}} \text{Im}[f_{11}(C)] \sin \theta \sin \phi$$

$$+ \sqrt{\frac{3}{4\pi}} f_{10}(C) \cos \theta + \cdots, \quad \text{(6.3)}$$
where Re and Im designate the real and imaginary parts. With reference to Fig. 4, it can be seen that

\[
\sin \theta \cos \phi = \frac{C_1}{C} \\
\sin \theta \sin \phi = \frac{C_2}{C}
\]

and

\[
\cos \theta = \frac{C_3}{C}.
\]

Equation (6.3) can therefore be written in the form

\[
f_e(C) = f^0(C) + f^1(C) \cdot \frac{C}{C} + \cdots (6.4)
\]

by means of the definitions

\[
f^0(C) = \frac{1}{\sqrt{4\pi}} f_{00}(C)
\]

and

\[
f^1(C) = \begin{cases} 
- \frac{3}{8\pi} \frac{2 \text{Re}[f_{11}(C)]}{2} \\
\frac{3}{8\pi} \frac{2 \text{Im}[f_{11}(C)]}{2} \\
\frac{3}{4\pi} f_{10}(C)
\end{cases}
\]

The foregoing discussion is intended to make plausible, without going further into the details of the spherical-harmonic expansion, the equivalence of the spherical-harmonic expansion (6.1) and the Cartesian-tensor expansion

\[
f_e(C) = f^0(C) + f^1(C) \cdot \frac{C}{C} + f^2(C) \cdot \frac{C^2}{C^2} + f^3(C) \cdot \frac{C^3}{C^3} + \cdots , (6.5)
\]

the first terms of which appear in equation (6.4). The expansion coefficients \(f^0, f^1, f^2, f^3, \ldots \) in equation (6.5) are real tensors of increasing order—\(f^0\) being a scalar, \(f^1\) a vector, \(f^2\) a second-order tensor, etc.—which depend on the electron speed \(C\) and, in general, position and time. For example, under certain conditions the vector \(f^1\) is given in terms of the
electric-field vector $E$ by $\mathbf{E} = a(C)E$, where $a(C)$ is a scalar function of $C$. Although the orthonormal properties of the spherical-harmonic functions can be used to advantage in the pertinent calculations, we shall not emphasize the mathematical details of these calculations. Since the electron Boltzmann equation (3.5) and the equations that we wish to derive in this section are expressed in terms of Cartesian tensors, we can most simply work with the Cartesian-tensor form (6.5) of the expansion. We note at this point that the expansion (6.5) contains as a special case the Chapman-Enskog expansion (Chapman and Cowling, 1952; and Hirschfelder, Curtiss, and Bird, 1964) for the electron distribution function. This will be made manifest in our discussion in the following chapter of transport properties.

It can be shown that for the functions $f_{\phi \phi}$, $f_{\phi \rho \rho}$, ... in the expansion (6.5) it is sufficient to consider only tensors that are unaffected by an interchange of any two indices and that sum to zero when any two indices are set equal. Such tensors are called symmetric irreducible tensors. For example, we then have

\[ f_{\phi \phi} = f_{\rho \rho}, \]
\[ f_{\phi \rho} = f_{\rho \phi} = f_{\phi \rho} = f_{\rho \phi} = f_{\phi \rho} = f_{\rho \phi} = 0, \]
\[ f_{\phi \rho \rho} = f_{\rho \phi \rho} = f_{\rho \rho \phi} = f_{\rho \phi \rho} = f_{\rho \rho \phi} = f_{\rho \phi \rho} = 0, \]

(6.6)

and

\[ f_{\phi \rho \rho \rho} = f_{\phi \phi \rho} = f_{\phi \phi \phi} = 0. \]

That this is so can be illustrated by a consideration of the second-order terms in the expansion (6.5). Since, for example, $C_1 C_2 = C_2 C_1$, the two terms $f_{11}^2 C_1 C_2 / C^2$ and $f_{22}^2 C_2 C_1 / C^2$ are clearly not independent and could be written as a single term. This is accomplished by the equality $f_{11}^2 = f_{22}^2$, indicated in the first of equations (6.6). Furthermore, since $C_1^2 / C^2 + C_2^2 / C^2 + C_3^2 / C^2 = 1$, the terms $f_0, f_{11}^1 C_1 / C^2, f_{22}^1 C_2 / C^2, f_{33}^1 C_3 / C^2$, and $f_{12}^3 C_3 / C^2$, are not linearly independent and, in effect, only three of these terms need be retained. This is in turn accomplished by the second of equations (6.6). Thus with the use of the condition $f_{\phi \rho} = 0$, the terms in question could be written as three linearly independent terms, for example, in the form

\[ f_0 + f_{11}^1 C_1 / C^2 + f_{22}^1 C_2 / C^2 + (-f_{11}^2 - f_{22}^2) \left(1 - \frac{C_1^2}{C^2} - \frac{C_2^2}{C^2}\right) \]

\[ = (f_0 - f_{11}^2 - f_{22}^2) + (2f_{11}^1 + f_{22}^1) \frac{C_1^2}{C^2} + (2f_{12}^3 + f_{33}^1) \frac{C_3^2}{C^2}. \]

It can be verified (see Jackson, 1962, Sec. 3.4) that with the conditions (6.6) and (6.2) the number (5) of independent second-order coefficients $f_{\phi \rho}$.
in the Cartesian-tensor expansion (6.5) is just equal to the number of
independent real and imaginary parts of the corresponding coefficients \( f_{2m} \)
in the spherical-harmonic form (6.1). Arguments similar to the foregoing
apply to the conditions (6.6) on the higher-order expansion coefficients such
as \( f_{3q} \). We will not be concerned in detail with such terms, however, since
they make no contribution to our final equations. In fact, all the results
of the following chapter could be obtained from the form (6.4) of the
expansion, with only the terms that involve \( f^0 \) and \( f^1 \) retained.

In principle, the Cartesian-tensor expansion (6.5) is valid for any
sufficiently well-behaved function of electron velocity. The practical utility
of this expansion, however, rests on the condition that departures of
\( f_e(C) \) from an isotropic distribution \( f^0(C) \) be small, that is, that
\( f^0 \gg |f^1|, |f^2|, |f^3|, \ldots \). (6.7)

An estimate of the range of applicability of the condition (6.7) can be
made from the definitions of Sec. 2 of the electron mean speed, diffusion
velocity, heat flux, pressure, and viscous stress tensor. When the expansion
(6.5) is used for \( f_e(C) \) in these definitions it is readily shown [see equations
(6.19), which follow, and Exercise 6.2] that
\[
C_e = \int_{-\infty}^{\infty} C f_e d^3C = 4\pi \int_0^\infty C^3 f^0 dC,
\]
\[
U_{se} = \int_{-\infty}^{\infty} C f_e d^3C = \frac{4\pi}{3} \int_0^\infty C^2 f^1 dC,
\]
\[
q_e = \frac{1}{2} n_e m_e \int_{-\infty}^{\infty} C_\alpha C^2 f_e d^3C = \frac{2\pi}{3} n_e m_e \int_0^\infty C^5 f^1 dC,
\]
\[
p_e = \frac{1}{3} n_e m_e \int_{-\infty}^{\infty} C^2 f_e d^3C = \frac{4\pi}{3} n_e m_e \int_0^\infty C^4 f^0 dC,
\]
and
\[
\tau_{e\alpha} = -\left[ n_e m_e \int_{-\infty}^{\infty} C_\alpha C \delta f_e d^3C - p_e \delta_{\alpha \beta} \right] = -\frac{8\pi}{15} n_e m_e \int_0^\infty C^4 f^2_{\alpha \beta} dC.
\]

We can then estimate that
\[
\frac{f^1}{f^0} \sim \max \left\{ \frac{U_{se} C_e}{q_e C_e p_e} \right\}
\]
and
\[
\frac{f^2}{f^0} \sim \frac{\tau_e}{p_e},
\]
(6.9)
where $f^1 = |f^1|, \tau_x \equiv |\tau_{xep}| = (\tau_{xe}, \tau_{ep})^{1/2}$, etc. Thus in a collision-dominated plasma for which

$$\frac{U_e}{C_e} \ll 1, \quad (6.10a)$$

$$\frac{q_e}{C_e P_e} \ll 1, \quad (6.10b)$$

and

$$\frac{\tau_e}{P_e} \ll 1 \quad (6.10c)$$

it is reasonable to seek a solution of the electron Boltzmann equation that satisfies the condition (6.7).

The functions $f^0, f^1, f^2, \ldots$ can be found from the equations that are formed by the substitution of the expansion (6.5) in the electron Boltzmann equation, followed by the successive multiplication of this equation by $1, C_{eh}/C_e, C_{ee}/C_e, \ldots$ and the integration of the resulting equations over solid angle in velocity space. We shall be interested in particular in the functions $f^0$ and $f^1$, which under certain conditions can be found from the first two of these equations.

From equation (3.5), the electron Boltzmann equation can be written symbolically as

$$\frac{\partial f_e}{\partial t} = \sum_h C_{eh} + C_{ee}, \quad (6.11a)$$

where

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial t} (n_e f_e) + (u_\beta + C_\beta) \frac{\partial}{\partial x_\beta} (n_e f_e) + \left( \frac{F_{e\beta}}{m_e} - \frac{Du_\beta}{Dt} \right) n_e \frac{\partial f_e}{\partial C_\beta} - C_\beta \frac{\partial u_\beta}{\partial x_\beta} n_e \frac{\partial f_e}{\partial C_\beta} \quad (6.11b)$$

and where the electron/heavy-particle and electron-electron collision terms $C_{eh}$ and $C_{ee}$ are given by equations (5.17) and (4.15), respectively. The first two of the Cartesian-tensor equations formed from the electron Boltzmann equation are then

$$\frac{1}{4\pi} \int_0^{4\pi} \frac{\partial f_e}{\partial t} \, d\Omega_C = \frac{1}{4\pi} \int_0^{4\pi} \left( \sum_h C_{eh} + C_{ee} \right) \, d\Omega_C \quad (6.12)$$

and

$$\frac{3}{4\pi} \int_0^{4\pi} C \frac{\partial f_e}{\partial t} \, d\Omega_C = \frac{3}{4\pi} \int_0^{4\pi} C \left( \sum_h C_{eh} + C_{ee} \right) \, d\Omega_C, \quad (6.13)$$
where \(d\Omega_c\) is a differential solid angle in velocity space. Division by the factors \(4\pi\) and \(4\pi/3\) has been introduced here for later convenience. The collision terms in equations (6.12) and (6.13) can be written more compactly by means of the definitions

\[
C_{eh}^0 = \frac{1}{4\pi} \int_0^{4\pi} C_{eh} \, d\Omega_c, \\
C_{ee}^0 = \frac{1}{4\pi} \int_0^{4\pi} C_{ee} \, d\Omega_c, \\
C_{eh}^1 = \frac{3}{4\pi} \int_0^{4\pi} \frac{C}{C} C_{eh} \, d\Omega_c, \\
C_{ee}^1 = \frac{3}{4\pi} \int_0^{4\pi} \frac{C}{C} C_{ee} \, d\Omega_c.
\]

and

\[
C_{ee}^0 = \frac{3}{4\pi} \int_0^{4\pi} \frac{C}{C} C_{ee} \, d\Omega_c.
\]

By reference to the discussion in Sec. 3 of the moments of the Boltzmann equation, it can be seen that equations (6.12) and (6.13) when multiplied respectively by \(\frac{1}{2}m_e C^2 \times 4\pi C^2 \, dC\) and \(m_e C \times (4\pi/3)C^2 \, dC\) and integrated over \(C\) yield the electron energy and momentum equations. Prior to the integration over \(C\), the equations thus formed are in fact speed-dependent energy and momentum equations for the electrons. That is, they express energy and momentum balances for the electrons in the speed range from \(C\) to \(C + dC\).

In the remainder of this section we discuss the evaluation of equations (6.12) and (6.13) when \(f_e(C)\) is expressed in terms of the Cartesian-tensor expansion (6.5). In these calculations we specify that the external force \(F_e\) on the electrons is given, as in equation (3.13), by

\[
F_e = -e(E' + C \times B),
\]

where

\[
E' = E + u \times B.
\]

Since \(F_e/m_e\) and \(Du/Dt\) appear together on the left-hand side (6.11b) of the Boltzmann equation, it is convenient to introduce the acceleration vector \(a'\), which is defined by

\[
a' = - \left( \frac{eE'}{m_e} + \frac{Du}{Dt} \right).
\]

We then have

\[
\frac{F_e}{m_e} \frac{Du}{Dt} = a' - \frac{e}{m_e} C \times B.
\]
We now consider the evaluation of equation (6.12). From equation (6.11b) for $\frac{Df_i}{Dt}$ and equations (6.19) which follow, the terms on the left-hand side of equation (6.12) are

$$\frac{1}{4\pi} \int_0^{4\pi} \frac{\partial}{\partial t} \left[ n_e \left( f^0 + \frac{C_\beta}{C} f_\beta + \cdots \right) \right] d\Omega_C = \frac{\partial}{\partial t} \left( n_e f^0 \right),$$

$$\frac{1}{4\pi} \int_0^{4\pi} u_\beta \frac{\partial}{\partial x_\beta} \left[ n_e \left( f^0 + \frac{C_\beta}{C} f_\beta + \cdots \right) \right] d\Omega_C = u_\beta \frac{\partial}{\partial x_\beta} \left( n_e f^0 \right),$$

$$\frac{1}{4\pi} \int_0^{4\pi} C_\beta \frac{\partial}{\partial x_\beta} \left[ n_e \left( f^0 + \frac{C_\beta}{C} f_\beta + \cdots \right) \right] d\Omega_C = \frac{1}{3} C \frac{\partial}{\partial x_\beta} \left( n_e f_\beta \right),$$

$$\frac{1}{4\pi} \int_0^{4\pi} \left( \frac{F_{\beta}}{m_e} - \frac{Du_\beta}{Dt} \right) n_e \frac{\partial}{\partial x_\beta} \left( f^0 + \frac{C_\beta}{C} f_\beta + \cdots \right) d\Omega_C = \frac{1}{3} a'_\beta n_e \frac{1}{C^2} \frac{\partial}{\partial C} \left( C^2 f_\beta \right),$$

and

$$-\frac{1}{4\pi} \int_0^{4\pi} \frac{\partial u_\beta}{\partial x_\beta} n_e \frac{\partial}{\partial C} \left( f^0 + \frac{C_\beta}{C} f_\beta + \cdots \right) d\Omega_C = -\frac{1}{3} C \frac{\partial u_\beta}{\partial x_\beta} n_e \frac{\partial}{\partial C} \left( C^2 f_\beta \right) - \frac{2}{15} \frac{\partial u_\beta}{\partial x_\beta} n_e \frac{\partial}{\partial C} \left( C^3 f_\beta \right).$$

Here we have used the following relations which hold for any scalar function $\Phi(C)$:

$$\int_0^{4\pi} \Phi \ d\Omega_C = 4\pi \Phi,$$

$$\int_0^{4\pi} C \Phi \ d\Omega_C = 0,$$

$$\int_0^{4\pi} C^2 \Phi \ d\Omega_C = \int_0^{4\pi} C^2 \Phi \ d\Omega_C = \int_0^{4\pi} C^2 \Phi \ d\Omega_C = \frac{4\pi}{3} C^2 \Phi,$$

$$\int_0^{4\pi} C_\beta C \Phi \ d\Omega_C = \frac{4\pi C^2}{3} \Phi \delta_{\alpha\beta},$$

and

$$\frac{\partial \Phi}{\partial C_\alpha} = \frac{C_\beta}{C} \frac{\partial \Phi}{\partial C}.$$

(These relations can be readily derived from a consideration of the spherical polar coordinates of Fig. 4, in terms of which $d\Omega_C = \sin \theta \ d\theta \ d\phi$, $C_1 = C \sin \theta \ cos \phi$, $C_2 = C \ sin \ \theta \ sin \ \phi$, and $C_3 = C \ cos \ \theta$.) From
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equations (6.18) and similar relations for higher-order tensors and with the properties such as (6.6) of symmetric irreducible tensors, it follows that

\[
\int_0^{4\pi} \Phi f_e d\Omega_c = 4\pi \Phi f^0,
\]

\[
\int_0^{4\pi} C\Phi f_e d\Omega_c = \frac{4\pi}{3} C\Phi f^1,
\]

\[
\int_0^{4\pi} \frac{\partial}{\partial C_\alpha} (\Phi f_e) d\Omega_c = \frac{4\pi}{3} \frac{1}{C^2} \frac{\partial}{\partial C} (C^2 \Phi f^1),
\]

\[
\int_0^{4\pi} C_s \frac{\partial}{\partial C_\beta} (\Phi f_e) d\Omega_c = \frac{4\pi}{3} C \frac{\partial}{\partial C} (\Phi f^0) \delta_{s\beta} + \frac{8\pi}{15} \frac{1}{C^2} \frac{\partial}{\partial C} (C^3 \Phi f^2),
\]

(6.19)

and also

\[
\int_0^{4\pi} C_s C_\beta \Phi f_e d\Omega_c = \frac{4\pi}{3} C^2 \Phi f^0 \delta_{s\beta} + \frac{8\pi}{15} C^2 \Phi f^2.
\]

Equations (6.17), and also the foregoing equations (6.8), can be obtained directly from equations (6.19).

Combining the results from equations (6.17), we have for the first Cartesian-tensor equation (6.12)

\[
\frac{D}{Dt} (n_e f^0) + \frac{C}{3} \frac{\partial}{\partial x_\beta} (n_e f^1) + \frac{a_{\beta\gamma} n_e}{3C^2} \frac{\partial}{\partial C} (C^2 f^1) = \frac{C}{3} \frac{\partial}{\partial x_\beta} n_e \frac{\partial f^0}{\partial C} - \frac{2}{15} \frac{\partial u_\gamma}{\partial x_\beta} \frac{1}{C^2} \frac{\partial}{\partial C} (C^3 f^2) = \sum_h C_{eh}^0 + C_{eh}^0, \quad (6.20)
\]

where \( C_{eh}^0 \) and \( C_{ee}^0 \) are defined by equations (6.14).

We next discuss the collision term

\[
C_{eh}^0 = \frac{1}{4\pi} \int_0^{4\pi} C_{eh} d\Omega_c.
\]

From the expansion for small electron mass (see Exercise 6.3), \( C_{eh} \) is given by equation (5.17) as

\[
C_{eh} = n_e \int_{4\pi} \left[ f_e(V) - f_e(C) \right] \hat{v}_{eh} d\Omega_k
\]

\[
+ n_e \int_{4\pi} \left[ f_e(V) - f_e(C) \right] \left[ -U_k \frac{\partial \hat{v}_{eh}}{\partial C_\beta} + \frac{1}{2} \frac{kT_h}{m_h} \frac{\partial^2 \hat{v}_{eh}}{\partial C_\beta \partial C_\gamma} \right] d\Omega_k
\]

\[
- 2n_e \int_{4\pi} k_{\beta} \frac{\partial f_e(V)}{\partial C_\beta} U_h \cdot \hat{k}_{eh} d\Omega_k
\]

\[
+ 2n_e \frac{m_e}{m_h} \frac{\partial}{\partial C_\beta} \left[ \int_{4\pi} k_{\beta} k_{\gamma} \left[ C_{\gamma} f_e(V) + \frac{kT_h}{m_h} \frac{\partial f_e(V)}{\partial C_\gamma} \right] \hat{v}_{eh} d\Omega_k \right]. \quad (6.21)
\]
It can be verified that the contribution to $C_{eh}$ from the first term on the right-hand side of equation (6.21) vanishes. This follows mathematically from the relations $V = C$ [equation (5.5)] and $dQ_C = dQ_v$ or alternatively from the properties of the expansion coefficients $f^0, f^1, \ldots$. Physically, $C_{eh}$ is proportional to the electron/heavy-particle collision term in the speed-dependent electron energy equation. Since the first term in equation (6.21) corresponds to the scattering of electrons by infinitely massive heavy particles without loss of energy, this term will not contribute to the electron energy equation.

We must therefore account for the higher-order terms in equation (6.21). In these terms, since we assume that $f^0 \gg f^1, f^2, \ldots$ [equation (6.7)], it is sufficient to consider only the contribution of $f^0$. Then, since $f^0(V) = f^0(C)$, the second integral in equation (6.21) vanishes immediately, and by equations (6.18) and (6.24), which follows, the third integral vanishes on integration over $dQ_C$. From the last integral we obtain

$$C_{eh} = \frac{1}{4\pi} \int \frac{m_e}{m_v} \frac{d}{dC} \left[ C_f f^0(C) + \frac{kT_b}{C} \frac{\partial f^0(C)}{\partial C} \right] \times 2 \int_{dQ_C} k_{\beta} k_{\gamma} \tilde{\nu}_{eh} dQ \, dQ_{C}. \quad (6.22)$$

To evaluate the right-hand side of equation (6.22), we consider first the integral

$$2 \int k_{\beta} k_{\gamma} C_{\beta} \tilde{\nu}_{eh} dQ_{C}. \quad (6.23)$$

With reference to Fig. 5, it can be seen that the apse vector $k$ can be written as the sum of two vectors, one $k_{\parallel} = C \cos \psi/C$ that is parallel to $C$ and the other $k_{\perp}$ that is perpendicular to $C$. In the integration of the expression (6.23) with respect to the azimuthal angle $\phi$, the contribution of $k_{\perp}$ vanishes. Since the dot product $k_{\beta} C_{\beta}$ is equal to $C \cos \psi$, the expression (6.23) then becomes

$$2 \int k_{\parallel} k_{\beta} C_{\beta} \tilde{\nu}_{eh} dQ_{C} = 2 \int k_{\parallel} C \cos \psi \tilde{\nu}_{eh} dQ_{C}$$

$$= 2C \int \cos^2 \psi \tilde{\nu}_{eh} dQ_{C}$$

$$= C \int (1 - \cos \chi) \tilde{\nu}_{eh} dQ_{C}$$

$$= C \nu_{eh}^{(1)}(C). \quad (6.24)$$
Here, by equations (5.6) and (5.12), the momentum-transfer collision frequency \( \nu_{\text{eh}}^{(1)}(C) \) is given by

\[
\nu_{\text{eh}}^{(1)}(C) \equiv \int_{\Omega_h} (1 - \cos \chi') \nu_{\text{eh}} \, d\Omega_k = n_h \int_{\Omega_h} (1 - \cos \chi') C I_{\text{eh}}(C, C \cdot k) \, d\Omega_k
\]

\[
= n_h \int_{\Omega_h} (1 - \cos \chi') C I_{\text{eh}}(C, \chi') \, d\Omega.
\] (6.25)

It follows that \( \nu_{\text{eh}}^{(1)}(C) \) is the same function as the collision frequency \( \nu_{\text{eh}}^{(1)}(g) \equiv n_h g Q_{\text{eh}}^{(1)}(g) \), where \( Q_{\text{eh}}^{(1)}(g) \) is defined by equations (II 3.7) and (4.9) (see Fig. 3 of Sec. 5).

With this result (6.24), equation (6.22) for \( C_{\text{eh}}^0 \) becomes

\[
C_{\text{eh}}^0 = \frac{1}{4\pi} n_e \frac{m_e}{m_h} \int_0^{4\pi} \frac{\partial}{\partial C} \left[ C g \left( f^0 + \frac{kT_e}{m_e} \frac{1}{C} \frac{\partial f^0}{\partial C} \right) \nu_{\text{eh}}^{(1)} \right] \, d\Omega_C
\]

\[
= n_e \frac{m_e}{m_h} \int_0^{4\pi} \frac{\partial}{\partial C} \left[ C^3 \left( f^0 + \frac{kT_e}{m_e} \frac{1}{C} \frac{\partial f^0}{\partial C} \right) \nu_{\text{eh}}^{(1)} \right].
\] (6.26)

In anticipation of our later application of this expression and in accord with the discussion of Chap. II, Sec. 7, we note that when \( f^0 \) is a Maxwellian function (2.2) of electron temperature \( T_e \), we have

\[
(1/C)(\partial f^0/\partial C) = -(m_e/kT_e)f^0,
\]
so that the right-hand side of equation (6.26) is then proportional to the energy-loss factor \( (m_e/m_h)(T_e - T_h) \).

In the evaluation of the electron-electron collision term \( C_{ee}^0 \), from the Fokker-Planck expression (4.15), it is sufficient to consider only the contribution of \( f^0(C) \) in the expansion (6.5) for \( f(C) \), since \( f^0 \approx f^1, f^2, \ldots \).

With this condition we have

\[
C_{ee}^0 = \frac{1}{4\pi} \int_0^{4\pi} C_{ee} d\Omega_C
\]

\[
= n_e^2 \frac{\Gamma_{ee}}{4\pi} \int_0^{4\pi} \left[ \frac{\partial}{\partial C_B} \left( f^0 \frac{\partial H^0}{\partial C_B} \right) + \frac{1}{2} \frac{\partial^2}{\partial C_B \partial C_C} \left( f^0 \frac{\partial G^0}{\partial C_B} \right) \right] d\Omega_C,
\]

where

\[
\Gamma_{ee} = 4\pi \left( \frac{e^2}{4\pi \epsilon_0 m_e} \right)^2 \ln \Lambda
\]

and where

\[
H^0(C) \equiv 2 \int_{-\infty}^{\infty} f^0(W) d^3 W
\]

and

\[
G^0(C) \equiv \int_{-\infty}^{\infty} g f^0(W) d^3 W.
\]

In the definitions (6.29) of \( H^0 \) and \( G^0 \), \( W \) is a dummy variable for the velocity of the second electron in a collision. To evaluate these expressions, we regard the velocity \( C \) of the first electron as fixed and, as shown in Fig. 6, define \( \phi \) as the angle between \( C \) and \( W \). We can then write

\[
g^2 = W^2 + C^2 - 2WC \cos \phi
\]

and

\[
d^3 W = d\Omega_w W^2 dW
= 2\pi \sin \phi \ d\phi W^2 dW.
\]

Figure 6. The angle \( \phi \).
With equation (6.30) for \( g \) it follows from direct integration that

\[
\int_0^\pi \frac{x \sin \phi}{g} \, d\phi = \int_0^{(W + C)} [(W + C) \pm (W - C)]
\]

and

\[
\int_0^\pi g \sin \phi \, d\phi = \frac{1}{3WC} [(W + C)^3 \pm (W - C)^3],
\]

where the positive sign applies if \( W < C \) and the negative sign applies if \( W > C \). Combining these expressions we have

\[
H^0 = 8\pi \left( \frac{1}{C} \int_0^C W^2 f^0(W) \, dW + \int_C^\infty W f^0(W) \, dW \right)
\]

and

\[
G^0 = 4\pi \left( \frac{1}{3C} \int_0^C W^4 f^0(W) \, dW + C \int_0^C W^2 f^0(W) \, dW \right.
\]

\[
+ \int_C^\infty W^3 f^0(W) \, dW + \frac{C^2}{3} \int_C^\infty W f^0(W) \, dW \right). (6.31)
\]

When equations (6.31) are substituted in equation (6.27) for \( C_{ee}^0 \), the result can be simplified by use of the fact that \( H^0 \) and \( G^0 \) depend on \( C \) only through the speed \( C \) and by use of the relations

\[
\frac{\partial^2 H^0}{\partial C_\phi \partial C_\phi} = \frac{\partial^2 G^0}{\partial C_\phi \partial C_\phi} = -8\pi f^0(C),
\]

which can be verified by direct calculation. After some manipulation, \( C_{ee}^0 \) can then be expressed in the form

\[
C_{ee}^0 = n_e^2 \Gamma_{ee} \frac{1}{C^2} \frac{\partial}{\partial C} \left[ I_0^0 f^0 + (I_0^0 + J_0^0) \frac{C}{3} \frac{\partial f^0}{\partial C} \right], \tag{6.32}
\]

where the integrals \( I_0^0 \) and \( J_0^0 \) are defined by

\[
I_0^0 \equiv \frac{4\pi}{C^2} \int_0^C C^2 + \gamma f^0 \, dC
\]

and

\[
J_0^0 \equiv \frac{4\pi}{C^2} \int_C^\infty C^2 + \gamma f^0 \, dC. \tag{6.33}
\]
In these definitions we have discontinued the use of the dummy variable \( W \) as no longer necessary. Equations (6.20), (6.26), and (6.32) can now be combined to yield the first (6.12) of the Cartesian-tensor equations.

The second equation that we require (6.13) is formed by the multiplication of the Boltzmann equation by \( 3C/4\pi C \) and integration over solid angle in velocity space. By means of an analysis that parallels the foregoing, the result is [see Shkarofsky, Johnston, and Bachynski (1966), Sec. 3.7],

\[
\frac{D}{Dt}(n_e f_e^1) + C \frac{\partial}{\partial x_a} (n_e f_0^e) + \alpha'_e n_e \frac{\partial f_0^e}{\partial C} - \frac{e}{m_e} B \times n_e f_e^1 \\
+ 2 \frac{C}{5} \frac{\partial}{\partial x_{\beta}} n_e f_{\beta}^2 + \frac{2}{5C^3} \frac{\partial}{\partial C} \left( C^3 \alpha'_e n_e f_{\beta}^2 \right) - \frac{\partial u_{\beta}}{\partial x_{\beta}} n_e f_{\beta}^1 \\
- \left( \frac{\partial u_{\beta}}{\partial x_{\alpha}} + \frac{\partial u_{\alpha}}{\partial x_{\beta}} \delta_{\alpha\beta} \right) \frac{C^2}{5} \frac{\partial}{\partial C} \left( f_{\beta}^1 \right) \\
- \frac{6}{35} \frac{\partial u_{\beta}}{\partial x_{\alpha}} n_e \frac{\partial}{\partial C} \left( C^4 f_{\alpha \beta} \right) = \sum_k \mathcal{C}_{eh}^k + \mathcal{C}_{eh}^l. \tag{6.34}
\]

The derivation of the electron/heavy-particle collision term

\[
\mathcal{C}_{eh} \equiv \frac{3}{4\pi} \int C \frac{f_0^e}{C} \mathcal{C}_{eh} d\Omega_c
\]

[equation (6.14c)] is similar to that of the term \( \mathcal{C}_{eh}^0 \). Again, since \( V = C \), the function \( f_0^e \) does not contribute to the first integral in equation (6.21) for \( \mathcal{C}_{eh} \). The only nonzero contribution to \( \mathcal{C}_{eh}^0 \) from this integral is that from \( f^1 \). With \( f^1(V) = f^1(C) \), this term is [see Fig. 5 and equations (6.25) and (6.18)]

\[
\frac{3n_e}{4\pi} \int_0^{4\pi} \int_{\Omega_c} \frac{C}{V} \left[ V \frac{f_0^e}{C} - \frac{C}{C} f_{\beta}^1 \right] \hat{V}_{eh} d\Omega_e d\Omega_C
\]

\[
= \frac{3n_e}{4\pi} \int_0^{4\pi} \int_{\Omega_c} \frac{C}{V} \left[ V - \frac{C}{C} f_{\beta}^1 \right] \hat{V}_{eh} d\Omega_e d\Omega_C
\]

\[
= \frac{3n_e}{4\pi} \int_0^{4\pi} \int_{\Omega_c} \left[ C \frac{C}{C} \hat{V}_{eh} \frac{f_{\beta}^1}{C} \right] \int_{\Omega_c} (\cos \chi' - 1) \hat{V}_{eh} d\Omega_e d\Omega_C
\]

In the remaining integrals in equation (6.21) for \( \mathcal{C}_{eh} \) only the contribution of \( f_0^e \) need be considered. Of these terms, with \( f_0^e = f^0 \), the first goes to zero immediately, and, after multiplication by \( 3C/4\pi C \) and integration over \( \Omega_c \), the last vanishes [cf. equation (6.26) and the second of equations
With the use of equations (6.24) and (6.18), the third term in equation (6.21) yields

\[
- \frac{3}{4\pi} 2n_e \int_0^{4\pi} \frac{C_\pi}{C} \int_{d\Omega_k} k_\beta \frac{\partial f}{\partial C'} U_h k_{hl} \tilde{v}_{eh} d\Omega_h d\Omega_c
\]

\[
= - \frac{3}{4\pi} n_e \int_0^{4\pi} \frac{C_\pi}{C} U_h \int_{d\Omega_k} 2k_\beta \frac{C_\beta}{C} \frac{\partial f}{\partial C'} \tilde{v}_{eh} d\Omega_h d\Omega_c
\]

\[
= - \frac{3}{4\pi} n_e \int_0^{4\pi} \frac{C_\pi}{C} U_h \frac{\partial f}{\partial C'} \tilde{v}_{eh}^{(1)} d\Omega_c
\]

\[
= - n_e U_h \tilde{v}_{eh}^{(1)} \delta_{\alpha \gamma} \frac{\partial f}{\partial C'}
\]

Combining these results, we have

\[
C_{eh} = - n_e \tilde{v}_{eh}^{(1)} \tilde{f}_1 - n_e \tilde{v}_{eh}^{(1)} U_h \frac{\partial f}{\partial C'}.
\]

We again refer to Shkarofsky, Johnston, and Brachynski (1966), Sec. 7.6, for the electron-electron collision term, which is

\[
C_{ee} = n_e^2 \Gamma_{ee} \left[ 8\pi f^{00} \tilde{f}_1 + \frac{1}{15C^2} (-3I_3^1 + 5I_1^1 + 2J_{-2}^1) \frac{\partial f}{\partial C'} \right.
\]

\[
+ \frac{1}{5C} (I_3^1 + J_{-2}^1) \frac{\partial^2 f}{\partial C^2}
\]

\[
+ \frac{1}{3C} \left( -I_2^0 + 3I_0^0 + 2J_0^0 \right) \left( - \frac{\tilde{f}_1}{C} + \frac{\partial \tilde{f}_1}{\partial C} \right)
\]

\[
+ \frac{1}{3C} \left( I_2^0 + J_0^0 \right) \frac{\partial^2 \tilde{f}_1}{\partial C^2} \right] ,
\]

where

\[
I_p^1 = \frac{4\pi}{C^p} \int_C C^{p+2} \tilde{f}_1^1 dC
\]

and

\[
J_p^1 = \frac{4\pi}{C^p} \int_C C^{p+2} \tilde{f}_1^1 dC.
\]
As indicated in Sec. 4, we regard electron-ion collisions as a particular class of collisions between electrons and heavy particles. The electron-ion contribution to equations (6.20) and (6.34) is then given by $C^{\theta}_{ee}$ and $C^{\gamma}_{ee}$ from equations (6.26) and (6.35) with $v^{(1)}_{eh}$ taken as the electron-ion momentum transfer collision frequency [see equation (4.10)]

$$v^{(1)}_{el} = n_e C^{(1)}_{el} (C) = \frac{4\pi n_i}{C^3} \left( \frac{Ze^2}{4\pi \varepsilon_0 m_e} \right)^2 \ln(\Lambda),$$

(6.37)

where $Z$ is the ion charge number. The same results can be obtained, but less directly, from the Cartesian-tensor expansion of the electron-ion Fokker-Planck collision term (4.15) by use of the simplifications that are appropriate to $m_i/m_j \ll 1$ (see Exercise 6.6).

Our derivation of the foregoing Cartesian-tensor equations (6.20) and (6.34) has been somewhat inconsistent in that we have applied the condition $f^0 \gg f^1$, $f^2$, ... to the collision terms, but not to the left-hand sides of equations (6.20) and (6.34). These equations can be simplified if we now apply this condition throughout. In addition, we assume also that the mass mean speed $u$ is no greater than the electron thermal speed $C_e$, that is, that

$$u \leq C_e.$$  

(6.38)

This inequality imposes a weak restriction on the Mach number $M$ of the flow. With $M \sim u/C_e \sim (u/C_e)(\bar{m}_h T_e/m_e T_e)^{1/2}$, we have $M \leq (\bar{m}_h T_e/m_e T_e)^{1/2}$, where $\bar{m}_h$ denotes a suitable average value of the heavy-particle mass. Under these conditions, we can omit certain terms on the left-hand sides of equations (6.20) and (6.34) in comparison with the terms that remain. These equations then become

$$\frac{D}{Dt} (n_e f^0) + \frac{C}{3} \nabla \cdot (n_e f^1) - \frac{e n_e}{3 m_e C^2} E^\prime \cdot \frac{\partial}{\partial C} (C^2 f^1) - \frac{C}{3} \nabla \cdot u n_e \frac{\partial f^0}{\partial C}$$

$$= n_e \sum_h \frac{m_e}{m_h} \frac{1}{C^2} \frac{\partial}{\partial C} \left[ C^3 \left( f^0 + \frac{k T_h}{m_e} \frac{1}{C} \frac{\partial f^0}{\partial C} \right) v^{(1)}_{eh} \right] + C^{\theta}_{ee}$$

(6.39)

and

$$\frac{\partial}{\partial t} (n_e f^1) + C \nabla (n_e f^0) + u n_e \frac{\partial f^0}{\partial C} - \frac{e}{m_e} B \times n_e f^1$$

$$= -n_e \sum_h v^{(1)}_{eh} \left( f^1 + U_h \frac{\partial f^0}{\partial C} \right) + C^{\gamma}_{ee},$$

(6.40)

where $C^{\theta}_{ee}$ and $C^{\gamma}_{ee}$ are given by equations (6.32) and (6.36), respectively. It can be seen that equations (6.39) and (6.40) comprise two equations for
the two unknown functions \( f^0 \) and \( f^1 \). Although these two equations are coupled, it is convenient, as we shall see in the following chapter, to regard the energy equation (6.39) as the equation which governs the isotropic part \( f^0 \) of the distribution function and the momentum equation (6.40) as the equation which governs \( f^1 \).

In spite of the imposing appearance of some of the equations of this section, this formulation provides a remarkable simplification in the calculation of the electronic transport properties of partially ionized plasmas. In Chapter VIII, we turn our attention to such calculations.

**Exercise 6.1.** With the use of equations (6.6), show that there are 5 and 7 independent components of \( f_{\epsilon \phi}^2 \) and \( f_{\phi \phi}^3 \), respectively.

**Exercise 6.2.** Derive equations (6.8) with the use of equations (6.19).

**Exercise 6.3.** Show that the viscous stress \( \tau_{\epsilon \phi} \), which was neglected in the derivation of equation (5.17) for \( C_{\epsilon \phi} \) from equation (5.16), would make no contribution to \( C_{\epsilon \phi}^0 \) and \( C_{\phi \phi}^0 \) as given by equations (6.26) and (6.35).

**Exercise 6.4.** Show that \( C_{\epsilon \epsilon}^0 = 0 \) when \( f^0 \) is given by the Maxwellian distribution (2.2).

**Exercise 6.5.** 1. Use the results of this section to obtain the electron energy and momentum equations that apply when \( f^0 \) is given by the Maxwellian distribution (2.2) and when \( U_\varepsilon = 0 \). Check your form of the energy equation against equation (VIII 3.16) of the following chapter.

2. Simplify the momentum equation for the case where \( v_{\epsilon \varepsilon}^{(1)} \) is independent of electron speed (the form \( v_{\epsilon \varepsilon}^{(1)} \neq v_{\phi \phi}^{(1)}(C) \) applies to so-called "Maxwellian molecules"). Compare your result with the "mean-free-path" Ohm's law (II 13.7b).

**Exercise 6.6.** For electron-ion collisions, derive equation (6.35) for \( C_{\epsilon \phi}^i \) from the Fokker-Planck collision term (4.15) for the case where \( U_\varepsilon = 0 \). In accord with the analysis that led to equation (6.35), you may simplify your calculations by considering only the zero-order approximation for \( m_e/m_i \ll 1 \) (so that \( g \approx C \)) and retaining only the terms in the Cartesian-tensor expansion (6.5) that involve \( f^0 \) and \( f^1 \).

**REFERENCES**


