impact parameter \( b \) and speed \( C \) relative to a bound electron transfers an amount of energy

\[
\frac{\pi^2 (e^2/4\pi\epsilon_0)^2}{2m_e C^3 b^2} \tilde{\gamma}^2 F(\tilde{\gamma}),
\]

where \( \tilde{\gamma} = \omega_b / C \) and where \( F(\tilde{\gamma}) \) is defined by equation (8.10a). (Apart from the absorption oscillator strength appearing as an additional factor, this expression is identical to the result obtained by Seaton, 1962, based on a quantum calculation, for the expected energy transfer in an electron-atom collision for a \( k \rightarrow l \) transition with \( \omega_k = \omega_l \). Seaton describes how this result may be used to obtain an expression for the electron-atom collisional excitation cross section.)

11. BLACKBODY RADIATION

An ideal blackbody is defined as one which absorbs all the radiant energy incident upon it. As shown in Fig. 12, an ideal blackbody can be approximated in practice as closely as desired by making a small hole in the side of a large hollow enclosure, the walls of which are kept at a constant temperature \( T \). A ray of radiation incident on the opening is reflected so many times on the inside surface before it eventually passes out of the hole that its energy is effectively fully absorbed. Towards the end of the nineteenth century, it was experimentally established that the spectral emission from a blackbody was a universal function of temperature alone, independent of the material of the

![Figure 12. An approximate blackbody source.](image-url)
walls. The blackbody radiation energy density spectrum in an enclosure $\mathcal{E}$, which is proportional to the emission, is illustrated schematically in Fig. 13.

The inadequacy of classical concepts to provide a satisfactory explanation of blackbody radiation was one of the earliest clues that led eventually to the development of quantum theory. Before going on to discuss blackbody radiation in quantum terms, it is necessary to derive the classical result which, as might be expected, is correct in a certain limit. Since the experimental evidence is that the spectral emission is independent of the wall material, we shall suppose the walls to be made up of

![Figure 13. Spectral distribution of blackbody radiation energy density.](image)
linear electronic harmonic oscillators. We would expect the result of the calculation to be independent of the parameters of the model, and this should be checked \textit{a posteriori}. The underlying notion of the derivation is that in order for a steady state to exist, the rate at which the walls absorb energy from the radiation field must be balanced by the rate at which the walls emit radiation. For linear electronic oscillators, these two processes are described by equations (10.23) and (10.5).

Equation (10.23) is an expression for the total energy absorbed by a linear electronic harmonic oscillator of natural frequency \( \omega_0 \) from a plane wave electromagnetic pulse of duration \( \tau \). Let us therefore assume that the radiation field in the enclosure is produced so as to last only a time \( \tau \) also. This assumption is merely a device that permits us to use the Fourier integral theorem simply, without getting involved in questions of convergence; the final physical results should not depend on \( \tau \). We can now relate the electric field factor \( |E_s(\omega_0)|^2 \) in equation (10.23) to the radiation energy density in an enclosure, as follows.

The radiation field may be regarded as a superposition of plane waves travelling in the three mutually perpendicular directions \( x \), \( y \), and \( z \). Using expression (4.5c) for the energy density in a plane wave, the energy density in the radiation field at any particular position, at some instant \( t \), is

\[
\mathcal{U}(t) = \epsilon_0 [E_x(t)^2 + E_y(t)^2 + E_z(t)^2].
\]

The average radiation energy density at this position, averaged over the time \( \tau \), is

\[
\overline{\mathcal{U}} = \epsilon_0 \left( \overline{E_x^2} + \overline{E_y^2} + \overline{E_z^2} \right).
\]

It is reasonable to assume that the time-averaged radiation field is isotropic and uniform, so that at all points in the enclosure,

\[
\overline{E_x^2} = \overline{E_y^2} = \overline{E_z^2}.
\]

Employing Parseval's theorem (see Footnote 6), we can therefore write

\[
\overline{\mathcal{U}} = \frac{3\epsilon_0}{\tau} \overline{E_y^2} = \frac{3\epsilon_0}{\tau} \int_{-\infty}^{\infty} E_y(t)^2 \, dt = \frac{12\pi\epsilon_0}{\tau} \int_0^{\infty} |E_s(\omega)|^2 \, d\omega = \int_0^{\infty} \mathcal{U}_\omega \, d\omega,
\]

where

\[
\mathcal{U}_\omega = \frac{12\pi\epsilon_0 |E_s(\omega)|^2}{\tau} \tag{11.1}
\]

is the radiation energy density per unit angular frequency.

Combining equations (11.1) and (10.23), we obtain

\[
\frac{W_{abs}}{\tau} = \frac{e^2}{12\epsilon_0 m_e} \mathcal{U}_\omega. \tag{11.2}
\]
In arriving at this expression, we have dropped the subscript zero in referring to the natural frequency of the oscillator and have used the relation $\nu_0 = 2\pi f_0$ for the radiation energy density per unit circular frequency. Equation (11.2) provides us with an expression for the average rate at which an electronic oscillator of natural frequency $v$ absorbs energy from a radiation field. We may note that this rate is independent of the total energy (or amplitude) of the oscillator.

The rate at which a particular oscillator of total energy $W$ and natural frequency $v$ radiates energy, averaged over a period, is given by equation (10.5) as

$$\bar{P} = \frac{2\pi e^2 v^2}{3\epsilon_0 m_e c^3} W.$$  \hspace{0.5cm} (11.3)

The oscillators comprising the enclosure wall will be distributed over various values of $W$. If we denote the average energy of the wall oscillators by $\langle W \rangle$, then the average rate at which oscillators of frequency $v$ radiate energy is

$$\langle \bar{P} \rangle = \frac{2\pi e^2 v^2}{3\epsilon_0 m_e c^3} \langle W \rangle.$$  \hspace{0.5cm} (11.4)

For steady state to exist between the walls and the radiation field, we must have

$$\frac{W_{\text{abs}}}{\tau} = \langle \bar{P} \rangle.$$  \hspace{0.5cm} (11.5)

From this condition, we obtain

$$\nu_0 = \frac{8\pi v^2}{c^3} \langle W \rangle.$$  \hspace{0.5cm} (11.6)

According to classical statistical mechanics, the average energy per degree of freedom of a system in thermodynamic equilibrium at a temperature $T$ is $kT/2$, where $k$ is Boltzmann's constant (see Vincenti and Kruger, 1965, p. 11). Since a linear harmonic oscillator has two degrees of freedom,

$$\langle W \rangle = kT.$$  \hspace{0.5cm} (11.7)

Employing equation (11.6), the condition for the radiation field to be in equilibrium with the enclosure walls is

$$\nu_0 = \frac{8\pi v^2 kT}{c^3}.$$  \hspace{0.5cm} (11.8)
This classical result for blackbody radiation, which can also be derived in other ways, was first obtained in 1900 and is known as the Rayleigh-Jeans formula. It is independent of the parameters of the oscillator model, as required; but as indicated in Fig. 13, it diverges for large frequencies. In 1901, Planck observed that an excellent fit to the experimental data could be obtained with equation (11.6) if one took for the average energy of an oscillator, instead of (11.7), the expression

\[ \langle W \rangle = \frac{h\nu}{\exp(h\nu/kT) - 1}. \]  

(11.9)

The constant \( h \) is obtained by fitting the formula for \( \langle W \rangle \) to the experimental results. Planck showed that the expression (11.9) for \( \langle W \rangle \) would result if one assumed that not all values of \( W \) are possible, as is the case for a classical oscillator, but that only the values \( W = n\hbar \nu \) are possible, where \( n = 0, 1, 2, \ldots \) (see Eisberg, 1961, pp. 63-66). We may note that if the gap \( \hbar \nu \) between successive energy levels is very much less than the thermal energy \( kT \), then the expression (11.9) reduces to the classical result (11.7).

In 1916, Einstein proposed a method for deriving the correct expression for the spectral distribution of blackbody radiation. The method remains essentially valid even within the framework of modern quantum theory. Let us consider a collection of atomic particles in a radiation field characterized by the specific intensity \( I_\nu \). The specific intensity is defined by equation (II 9.6) and is related to the radiation energy density spectrum by the equation (see Exercise II 9.1)

\[ \mathcal{W}_\nu = \frac{1}{c} \int_{4\pi} I_\nu(\Omega) \, d\Omega, \]  

(11.10)

where \( d\Omega \) is an element of solid angle about the direction \( \Omega \). For these particles, let us consider some particular frequency \( \nu = \nu_{lk} \equiv (\epsilon_l - \epsilon_k)/h \) corresponding, according to Bohr’s frequency assumption, to a transition between two energy levels \( \epsilon_l \) and \( \epsilon_k \). The situation is illustrated schematically in Fig. 14.

Einstein proposed the existence of three atomic constants to describe the interaction of the radiation with the matter. In accord with equation (9.6), the probability per unit time of spontaneous emission into the solid angle \( d\Omega \) about the direction \( \Omega \), corresponding to a transition from level \( l \) to level \( k \) is

\[ A_{lk} \frac{d\Omega}{4\pi}. \]  

(11.11a)
Figure 14. Einstein’s model for the derivation of the blackbody radiation spectrum.

The probability per unit time of induced emission into the solid angle \( d\Omega \), corresponding to an \( \ell \) to \( k \) transition is

\[
B_{\ell k} I_{\nu \ell} d\Omega. \tag{11.11b}
\]

Similarly, the probability per unit time of absorption from radiation propagating in the solid angle \( d\Omega \) about the direction \( \Omega \), accompanied by a \( k \) to \( l \) transition, is

\[
B_{kl} I_{\nu l} d\Omega. \tag{11.11c}
\]

If there are \( n_\ell \) and \( n_k \) particles per unit volume in levels \( \ell \) and \( k \), respectively, then the condition for a steady state is

\[
n_\ell(A_{\ell k}/4\pi + B_{\ell k} I_{\nu \ell}) = n_k B_{kl} I_{\nu l}. \tag{11.12}
\]

This equation is the quantum analog of the classical equation (11.5). Solving for \( I_{\nu l} \), we obtain

\[
I_{\nu l} = \frac{A_{\ell k}/4\pi B_{\ell k}}{(B_{kl}/B_{\ell k})(n_k/n_\ell) - 1}. \tag{11.13}
\]

Let us now assume that the collection of particles is in thermodynamic equilibrium. We may then employ the Boltzmann relation (II 10.1) for the ratio \( n_k/n_\ell \) and write

\[
I_{\nu l} = \frac{A_{\ell k}/4\pi B_{\ell k}}{(g_k B_{kl}/g_\ell B_{\ell k}) \exp(hv_{\ell k}/kT) - 1} \tag{11.13}
\]

as the condition for the radiation field to be in equilibrium with the matter. In the limit where the energy gap between the levels \( l \) and \( k \) is much less than the thermal energy \( kT \), the value of \( I_{\nu l} \) is given correctly by classical theory. For an isotropic radiation field

\[
I_{\nu l} = \frac{c}{4\pi} q_{\nu l}, \tag{11.14}
\]
so that for \( hv_{kl}/kT \ll 1 \), equations (11.8) and (11.13) yield

\[
\frac{2v_{kl}^2}{c^2}kT = \frac{A_{lk}/4\pi B_{lk}}{(g_k B_{kl}/g_l B_{lk})(1 + hv_{kl}/kT) - 1}.
\]

For this equation to hold for all values of \( kT \), consistent with the aforementioned inequality, it is necessary that

\[
g_k B_{kl} = g_l B_{lk}, \quad (11.15a)
\]

and that

\[
A_{lk} = \frac{8\pi hv_{kl}^3}{c^2} B_{lk}, \quad (11.15b)
\]

Substituting these relations back into equation (11.13) and writing \( v \) for \( v_{kl} \), the equilibrium spectral distribution of specific intensity \( B_v \), for any temperature, is

\[
B_v = \frac{2hv^3/c^2}{\exp(hv/kT) - 1}. \quad (11.16)
\]

At one stroke, Einstein's derivation\(^9\) leads not only to the blackbody radiation spectrum, but also to the important relations (11.15) between the \( A \) and \( B \) coefficients, showing that only one of these quantities is independent. The oscillator strength can also be related to the Einstein coefficients, since expressions (11.11c) and (10.20) are both descriptions of the absorption process. On the one hand, from equation (11.11c), the energy absorbed per unit time in all \( k \) to \( l \) transitions from radiation traveling in the solid angle \( d\Omega \) about \( \Omega \), is

\[
hv_{kl} B_{kl} I_{vl} d\Omega. \quad (11.17a)
\]

On the other hand, from equation (10.20), this same quantity is given by the expression

\[
\int_{\text{line}} (I_v d\Omega) Q_v^{(v-n)} dv.
\]

Assuming that \( I_v \) is a slowly varying function of frequency with respect to the line width, the latter expression can be written

\[
(I_{vl} d\Omega) \int_{\text{line}} Q_v^{(v-n)} dv = \frac{e^2f_{kl}}{4\epsilon_0 m_e c} I_{vl} d\Omega. \quad (11.17b)
\]

\(^9\) The application of the method of detailed balancing leading to equations (11.7) is essentially equivalent to the approach employed by Einstein.
Comparing equations (11.17), we obtain the result

\[ B_{kl} = \frac{e^2}{4\epsilon_0 m_e c h} \gamma_{ul} f_{ul} \]  

(11.18)

**Exercise 11.1.** At what approximate temperature is the energy density in an equilibrium radiation field equal to the random thermal energy in a gas at standard number density?

**Exercise 11.2.** Derive the relation \( A_{ik} = 3\gamma(g_k/g_l)f_{ul} \), where \( \gamma^{-1} \) is the natural lifetime of a classical electronic oscillator.

**Exercise 11.3.** Starting with the expression (11.17b), determine the total rate of energy absorption in an atomic transition from radiation traveling in all directions. Compare this result with the classical expression (11.2) for the rate of energy absorption by a classical linear harmonic oscillator. Explain any differences.

**REFERENCES**


