Table 1 Summary of Classical and Quantum Theory
Cross Section Properties

<table>
<thead>
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<th>Classical mechanics</th>
<th>Quantum mechanics</th>
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</table>
| $\mathcal{Q}(g)$ | 1. Infinite for all $\alpha$ | 1. Finite for $\alpha > 2$
|                |                     | 2. $\propto g^{-2(\alpha-1)}$ for $\alpha > 3$
|                |                     | and attraction |
| $\mathcal{Q}^{(1)}(g)$ | 1. Finite for $\alpha > 1$ | 1. Finite for $\alpha > 1$
|                | 2. Infinite for $\alpha = 1$ (Coulomb case) | 2. Infinite for $\alpha = 1$
|                | 3. $\propto g^{-4\alpha}$ for $\alpha > 1$ | (Coulomb case) |

At higher energies, elastic electron-atom scattering cross-section data are in accord with this result for many atoms (e.g., H$_2$ and He—see Fig. II 20). For such interactions, the momentum transfer collision frequency $v^{(1)} \propto g\mathcal{Q}^{(1)}$ is constant. The rigorous kinetic theory is uniquely simple for particles that have a constant momentum transfer collision frequency, and such particles are referred to as Maxwellian.

7. DOMAIN OF VALIDITY OF CLASSICAL THEORY

According to the Heisenberg uncertainty principle, simultaneous measurements of the position $x$ and momentum $p_x$ of a particle are possible only to the extent that the corresponding uncertainties $\Delta x$ and $\Delta p_x$ satisfy the inequality

$$\Delta x \Delta p_x \geq \hbar.$$ (7.1)

Here $\hbar = 2\pi\hbar$ is Planck's constant. Insofar as the very concept of a classical trajectory explicitly assumes the contrary, the classical description of a collision is in direct contradiction to quantum mechanics. However, we may use this principle to establish when classical mechanics provides a good approximation by determining the conditions under which it is possible to talk about an approximate trajectory within the limitations imposed by the condition (7.1).

For a classical description of a collision in terms of a trajectory to be meaningful in the face of uncertainty, it is necessary that two conditions be fulfilled. First, as illustrated in Fig. 10, the uncertainty in the initial direction of the particle must be small compared to the scattering angle, or

$$\frac{\Delta p_x}{p} \ll \chi,$$ (7.2)
where $p = m_1 g$ is the initial directed momentum of the particle. Second, the uncertainty in the impact parameter $\Delta b = \Delta x$ must be small compared to the impact parameter itself, or

$$\Delta b \ll b. \quad (7.3)$$

Combining equations (7.2) and (7.3) and employing (7.1), we must have

$$pb \chi \gg \Delta x \Delta p_x \gg \hbar,$$

or

$$b \chi \gg \frac{\hbar}{p} = \frac{\hbar}{m_1 g} \quad (7.4)$$

Let us apply this criterion to the results obtained in the preceding section for small-angle scattering by potentials $\phi(r) = \pm \frac{A}{r^\alpha}$ ($\alpha > 1$). Employing equation (6.6), we find that the classical calculation is valid provided

$$\frac{2Ah(\alpha)}{gb^{\alpha-1}} \gg \hbar. \quad (7.5)$$

This condition is always violated for sufficiently large impact parameters. We may therefore conclude that for the above potentials, classical collision theory is not capable of providing an acceptable description of scattering at very small angles.

Let us next consider the case of Coulomb scattering. For small scattering angles we obtain from equation (5.10)

$$\chi \approx \frac{2b_0}{b},$$
and therefore the criterion for the validity of the classical calculation is that

$$2b_0 \gg \frac{\hbar}{m_{12} g}.$$  (7.6)

But from equations (5.4) and (5.2)

$$b_0 = \frac{Ze^2}{4\pi\varepsilon_0 m_{12} g^2},$$  (7.7)

Substituting this expression for $b_0$ into (7.6) and introducing the velocity of light $c$, we may conclude that the classical calculation of the Rutherford differential scattering cross section is valid provided

$$\frac{g}{c} \ll 2Z \left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right) = \frac{2Z}{137}. \quad (7.8)$$

[The quantity in the brackets in equation (7.8) is the fine structure constant—cf. equation (II 4.6).] Thus this criterion is approximately equivalent to the condition that the relative speed be nonrelativistic.

We have seen that the momentum transfer collision cross section attributes a zero weight to small-angle scattering events. The breakdown of classical theory at small angles is therefore not too important, as far as calculations of $Q^{(1)}$ are concerned, and it is convenient to express the criterion (7.4) in another form. For impact parameters approximately equal to the range of the interaction $b \approx a \approx \sqrt{Q^{(1)}}$, the scattering angle is of order unity, and the criterion (7.4) may be written

$$\sqrt{Q^{(1)}} \gg \frac{\hbar}{m_{12} g}.$$  (7.9)

The right-hand side of this inequality is just the de Broglie wavelength of the reduced mass particle (except for an extra factor of $1/2\pi$), and in wave terms, this is just the condition that diffraction effects be small. This inequality is generally satisfied in collisions involving heavy particles, but it is not necessarily met for collisions involving electrons. Evidence of this behavior is exhibited in the elastic cross section data presented in Sec. II 14.

**Exercise 7.1.** Calculate the de Broglie wavelengths for the particles and conditions of Exercise II 6.6. Comment on the applicability of the criterion (7.9).