8. THE GENERALIZED OHM'S LAW

The current density $\mathbf{J}$ in a conducting fluid depends in general not only on the electric field $\mathbf{E}'$, as is usually the case for solids, but depends also on various other fluid properties. An equation describing this dependence is referred to as a "generalized Ohm's law." Equations (3.21) provide an example of such a relation, applicable to weakly ionized plasmas. In this section we wish to discuss a more general approach to the derivation of this type of relation.

We shall adopt the same continuum description as introduced in Sec. 5. Each species in the plasma is regarded as a separate fluid coexisting with the fluids made up of other species. The continuum equations of motion for each fluid are then similar in structure to the global equations of motion (4.1), except for additional terms that describe the interactions that a single-species fluid experiences in moving through the other separate fluids. In Sec. 5 we were interested in the energy equation, but for our present purposes we need to focus on the momentum equation.

In accordance with equation (4.1b), the momentum equation for the species $s$ may be written

$$\frac{\partial}{\partial t} (\rho_s \mathbf{u}_s) + \nabla \cdot (\rho_s \mathbf{u}_s \mathbf{u}_s) = -\nabla \cdot \mathbf{p}_s^* + e_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - \mathbf{M}_s. \quad (8.1)$$

Here $\rho_s = m_s n_s$ is the mass density of species $s$, and $\mathbf{u}_s$ is the fluid velocity of species $s$ defined in accordance with equation (II 6.2). The electromagnetic body-force term $e_s n_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$ is obtained by averaging equation (1.1), and it must be included if particles of species $s$ carry a charge $e_s$. From the point of view of kinetic theory (see Chapter VII), the species $s$ pressure tensor $\mathbf{p}_s^*$ is given in terms of an average over particle velocities as

$$\mathbf{p}_s^* \equiv \rho_s \int (\mathbf{c} - \mathbf{u}_s)(\mathbf{c} - \mathbf{u}_s)f_s(c)d^3c = \rho_s(c - \mathbf{u}_s)(c - \mathbf{u}_s).$$

For collision-dominated gases, it is conventional to employ instead of $\mathbf{p}_s^*$ the quantity [see equation (VII 2.6b)]

$$\mathbf{p}_s \equiv \rho_s(c - \mathbf{u})(c - \mathbf{u}),$$

$$= \rho_s(c - \mathbf{u}_s + \mathbf{U}_s)(c - \mathbf{u}_s + \mathbf{U}_s) = \mathbf{p}_s^* + \rho_s \mathbf{U}_s \mathbf{U}_s. \quad (8.2)$$
Here $U_s = u_s - u$ is the diffusion velocity of species $s$. To within the approximations we shall make, there is no difference between $p_s^*$ and $p_s$.

The pressure tensor for the gas as a whole is

$$p = \sum_s p_s.$$  \hspace{1cm} (8.3)

The interaction term $\mathcal{M}_s$ in equation (8.1) represents the average rate of momentum lost per unit volume by the species $s$ fluid as a result of collisions between particles of species $s$ with other particles [cf. equation (VII 3.10b)]. The rigorous treatment of this term requires, in effect, a kinetic theory approach to the problem as described in Chapter VII. We can, however, obtain useful approximate results more simply by employing the expression (II 7.16) for the average momentum lost per elastic collision between two types of particles. Thus, the average rate of loss of momentum per unit volume by $s$-type particles as a result of elastic collisions with $r$-type particles is

$$\mathcal{M}_{sr} \simeq n_s \bar{v}_{sr} m_{sr}(U_s - U_r).$$  \hspace{1cm} (8.4)

Here $m_{sr} = m_s m_r / (m_s + m_r)$ is the reduced mass and $\bar{v}_{sr} = n_r \bar{g}_{sr} \bar{Q}_{sr}$ is the average momentum transfer collision frequency between an $s$-type particle and $r$-type particles, as defined by equation (II 6.29).

In our subsequent discussion we shall, for simplicity, consider a three-species plasma consisting of electrons, singly charged positive ions, and neutrals. Employing the preceding equations, the momentum equations for these three fluids are

$$\frac{\partial}{\partial t} n_e u_e + \nabla \cdot m_e n_e (u_e u_e - U_e U_e)$$

$$= -e n_e (E' + U_e \times B) - \nabla \cdot p_e - n_e m_e \bar{v}_{ei}(U_e - U_i)$$

$$- n_e m_e \bar{v}_{en}(U_e - U_n),$$  \hspace{1cm} (8.5a)

$$\frac{\partial}{\partial t} m_i n_i u_i + \nabla \cdot m_i n_i (u_i u_i - U_i U_i)$$

$$= e n_i (E' + U_i \times B) - \nabla \cdot p_i - n_i m_i \bar{v}_{ei}(U_i - U_e)$$

$$- n_i m_i \bar{v}_{in}(U_i - U_n),$$  \hspace{1cm} (8.5b)

$$\frac{\partial}{\partial t} m_n n_n u_n + \nabla \cdot m_n n_n (u_n u_n - U_n U_n)$$

$$= -\nabla \cdot p_n - n_n m_e \bar{v}_{ne}(U_n - U_e) - n_n m_n \bar{v}_{ni}(U_n - U_i).$$  \hspace{1cm} (8.5c)

These equations are not in a very useful working form since they involve explicitly the number densities and diffusion velocities of the charged...
particles, rather than the charge and current densities

$$\rho^c = e(n_i - n_e),$$

(8.6a)

and

$$\mathbf{J} = e(n_i \mathbf{U}_i - n_e \mathbf{U}_e),$$

(8.6b)

which appear in the electrodynamic equations. In what follows we shall be concerned with how we can obtain more useful working relations from equations (8.5). To begin, we note that if these three equations are added the interaction terms cancel, and we obtain the global momentum equation [cf. equations (4.1b) and (4.4)]

$$\frac{\partial}{\partial t} \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{uu}) = -\nabla \cdot \mathbf{P} + \rho^c \mathbf{E}' + \mathbf{J} \times \mathbf{B}.$$  

(8.7)

In deriving this result we have used the relations $n_s \tilde{\mathbf{\nu}}_{sr} = n_r \tilde{\mathbf{\nu}}_{rs}$, $\rho \mathbf{u} = \sum_s \rho_s \mathbf{u}_s$, $\sum_s n_s \mathbf{u}_s \mathbf{u}_s = \rho \mathbf{uu} + \sum_s \rho_s \mathbf{U}_s \mathbf{U}_s$, and the fact that [see equation (116.7)]

$$\rho_e \mathbf{U}_e + \rho_i \mathbf{U}_i + \rho_n \mathbf{U}_n = 0.$$  

(8.8)

Let us first discuss the generalized Ohm's law for a fully ionized gas, for which the algebra is less involved. For this case, the last terms in equations (8.5a), (8.5b), and (8.8) are absent, and equation (8.5c) is not relevant. Multiplying equation (8.5a) by $-e/m_e$ and equation (8.5b) by $e/m_i$ and adding, one obtains, after some algebra, the result

$$\frac{m_e m_i}{e^2 \rho} \left[ \frac{\partial}{\partial t} (\rho^c \mathbf{u} + \mathbf{J}) + \nabla \cdot (\rho^c \mathbf{uu} + \mathbf{uJ} + \mathbf{J} \mathbf{u}) \right]$$

$$= \mathbf{E}' - \frac{m_e \tilde{\mathbf{\nu}}_{ei}}{n_i e^2} \mathbf{J} - \frac{(m_i - m_e)}{ep} (\rho^c \mathbf{E}' + \mathbf{J} \times \mathbf{B}) + \frac{\nabla \cdot (\rho_i \mathbf{p}_e - m_e \mathbf{p}_i)}{ep}.$$  

(8.9)

Equations (8.7) and (8.9) are entirely equivalent to equations (8.5a) and (8.5b). In effect, we have transformed from equations for $\mathbf{u}_e$ and $\mathbf{u}_i$ to equations for $\mathbf{u}$ and $\mathbf{J}$. From equation (8.6a) and the definition $\rho = m_e n_e + m_i n_i$ we can obtain $n_e$ and $n_i$ in terms of $\rho$ and $\rho^c$, while from equations (8.6b) and (8.8) we can obtain $\mathbf{U}_e$ and $\mathbf{U}_i$ in terms of $\mathbf{u}$ and $\mathbf{J}$. Thus,

$$n_e = \frac{ep - m_i \rho^c}{e(m_e + m_i)}, \quad n_i = \frac{ep + m_e \rho^c}{e(m_e + m_i)},$$

(8.10a)

and

$$\mathbf{U}_e = \frac{-m_i \mathbf{J}}{n_e (m_e + m_i)}, \quad \mathbf{U}_i = \frac{m_e \mathbf{J}}{n_i (m_e + m_i)}.$$  

(8.10b)

These relations are useful in deriving the result (8.9). For an electrically neutral plasma which is steady and uniform and with no $B$ field, equation (8.9) reduces to $\mathbf{J} = \sigma \mathbf{E}'$, where $\sigma = n_e e^2/m_e \tilde{\mathbf{\nu}}_{ei}$. It is for this reason that equation (8.9) is called a "generalized Ohm's law."
Equation (8.9) can be written in a simpler form applicable to most conditions of interest. Let us assume first that we are dealing with a plasma which satisfies the charge-neutrality condition

$$\frac{\rho^e}{en_e} \ll 1,$$  \hspace{1cm} (8.11)

so that $n_e \simeq n_i$, and that the characteristic time $t_c$ for macroscopic change is sufficiently large that [cf. equation (6.5b)]

$$t_c \gg \frac{\bar{v}_{ei}}{\omega_p^2}.$$  \hspace{1cm} (8.12)

Here $\omega_p = (n_e e^2/\varepsilon_0 m_e)^{1/2}$ is the plasma frequency. (For a steady-state situation, the time $t_c$ should be interpreted in terms of a characteristic distance $L_c$ and velocity $u_c$ as $t_c \sim L_c/u_c$.) In accordance with equations (6.8) and (6.9), the terms involving the factor $\rho^e$ in equation (8.9) can then be dropped.

If we assume also that

$$t_c \gg \bar{v}_{ei}^{-1},$$  \hspace{1cm} (8.13)

then all the terms involving $\mathbf{J}$ on the left-hand side of equation (8.9) are small compared to the term $\mathbf{J}/\sigma$ on the right-hand side, and the left-hand side of the equation can be replaced by zero.

Estimates of the magnitudes of the electron and ion viscous stresses may be obtained from the relations developed in Sec. II 12. It can be shown that

$$\left|\tau_e\right| \sim \frac{\bar{v}_{ei}^{-1}}{p_e} t_c,$$  \hspace{1cm} (8.14a)

and that

$$\frac{m_e}{m_i} \left|\tau_i\right| \sim \left(\frac{m_e T_e}{m_i T_i}\right)^{1/2} \frac{\bar{v}_{ei}^{-1}}{t_c}.$$  \hspace{1cm} (8.14b)

Thus for the conditions (8.11), (8.12), and (8.13), the generalized Ohm's law (8.9) for a fully ionized plasma can be written in the simpler form [cf. equation (3.21)]

$$\mathbf{E}' + \frac{\nabla p_e}{en_e} = \frac{\mathbf{J}}{\sigma} + \frac{\mathbf{J} \times \mathbf{B}}{en_e} = \frac{\mathbf{J} + \beta_e \mathbf{J} \times \mathbf{b}}{\sigma}.$$  \hspace{1cm} (8.15)

Here $\sigma = n_e e^2/m_e \bar{v}_{ei}$, $\beta_e = eB/m_e \bar{v}_{ei}$, and $\mathbf{b} = \mathbf{B}/B$.  

Let us turn our attention now to the full set of equations (8.5), for which the degree of ionization is arbitrary. Instead of equations (8.10) we now have

\[
\begin{align*}
n_e &= \frac{e(\rho_e + \rho_i) - m_i \rho_e}{e(m_e + m_i)}, \\
n_i &= \frac{e(\rho_e + \rho_i) + m_e \rho_e}{e(m_e + m_i)},
\end{align*}
\]

and

\[
\begin{align*}
U_e &= \frac{-m_i J - e \rho_n U_n}{en_e(m_e + m_i)}, \\
U_i &= \frac{m_e J - e \rho_n U_n}{en_i(m_e + m_i)}.
\end{align*}
\]

Multiplying equation (8.5a) by \(-e/m_e\) and equation (8.5b) by \(e/m_i\) and adding, one obtains in place of equation (8.9) the result

\[
\begin{align*}
\frac{m_e m_i}{e^2(\rho_e + \rho_i)} \left[ \frac{\partial}{\partial t} \left( \rho \mathbf{u} + \mathbf{J} \right) + \nabla \cdot (\rho \mathbf{u} \mathbf{u} + \mathbf{u} \mathbf{J} + \mathbf{J} \mathbf{u}) \right]
&= E' - \left[ \frac{m_e \bar{v}_{ei}}{n_i e^2} + \frac{m_i m_e}{e^2(\rho_e + \rho_i)} \left( \bar{v}_{en} + \frac{m_e m_{in}}{m_i} \bar{v}_{in} \right) \right] \mathbf{J}
- \frac{(m_i - m_e)}{e(\rho_e + \rho_i)} (\rho^c E' + \mathbf{J} \times \mathbf{B}) + \frac{\nabla \cdot (m_i \rho_e - m_e \rho_i)}{e(\rho_e + \rho_i)}

- \rho^c \frac{m_e \bar{v}_{ei}}{n_i e^2 (\rho_e + \rho_i)} - \frac{\rho_n}{\rho_e + \rho_i} \mathbf{U}_n \times \mathbf{B} - \frac{\rho}{\rho_e + \rho_i}

\times \frac{m_e}{e} \left[ \left( \bar{v}_{en} - \frac{m_{in}}{m_i} \bar{v}_{in} \right) - \rho^c \frac{m_i}{\rho e} \left( \bar{v}_{en} + \frac{m_e m_{in}}{m_i} \bar{v}_{in} \right) \right] \mathbf{U}_n. \tag{8.17}
\end{align*}
\]

Assuming the plasma satisfies the charge-neutrality condition (8.11) and noting that \(m_e \ll m_i\), the term involving the factor \(\mathbf{J}\) on the right-hand side of equation (8.17) becomes \(\mathbf{J}/\sigma\), where now \(\sigma = n_e e^2/m_e \bar{v}_{eh}\) and \(\bar{v}_{eh} = \bar{v}_{ei} + \bar{v}_{en}\). If we assume, corresponding to condition (8.12), that

\[
t_e \gg \bar{v}_{eh}/\omega_p^2, \tag{8.18}
\]

then the terms involving \(\rho^c\) can be dropped, and if, corresponding to condition (8.13)

\[
t_e \gg \bar{v}_{eh}^{-1}, \tag{8.19}
\]

the left-hand side of equation (8.17) can be replaced by zero. Corresponding to the estimates (8.14), we now have that

\[
\left| \frac{\tau_e}{\tau} \right| \sim \frac{\bar{v}_{eh}^{-1}}{\rho_e t_e}, \tag{8.20a}
\]
and that
\[
\frac{m_e |\tau|}{m_i p_e} \sim \frac{m_e T}{m_i T_e} \left( \bar{v}_{en} + \sqrt{\frac{m_e T}{m_i T_e} \bar{v}_{ie}} \right)^{-1}.
\] (8.20b)

Thus, for the right-hand side of (8.20b) small and for the conditions (8.11), (8.18), and (8.19), equation (8.17) reduces to the expression
\[
E' + \frac{\nabla p_e}{e n_e} = \frac{\mathbf{J} \times \mathbf{B}}{\sigma} + \frac{m_e \bar{v}_{en}}{e} \frac{\rho U_n}{\rho_i} - \mathbf{B} \times \frac{\rho n U_n}{\rho_i}.
\] (8.21)

Among the four equations (8.5) and (8.7), there are three independent relations. Thus, equations (8.17) [or (8.21)], (8.5c), and (8.7) may be viewed as providing three independent relations that are entirely equivalent to equations (8.5). To complete our derivation of the generalized Ohm's law, we need to obtain an expression for \(U_n\) from the two equations (8.5c) and (8.7), and then substitute this expression into equation (8.21). Employing the relations (8.16b) and the assumptions previously made in arriving at equation (8.21), we can write equation (8.5c) as follows:
\[
\rho_n \frac{D u}{D t} + \left[ \frac{D}{D t} \left( \rho_n U_n \right) + \rho_n U_n \cdot \nabla u + \rho_n U_n \nabla \cdot u \right] + \mathbf{u} \rho_n
\]
\[
= -\nabla \cdot p_n - \rho \frac{m_{in}}{m_i} \bar{v}_{in} U_n - \frac{m_e \bar{v}_{en}}{e} J. \quad (8.22)
\]

The second term on the left-hand side of this equation is of order \(\rho_n U_n / t_c\), and can be neglected in comparison with the second term on the right-hand side if
\[
t_c \gg (\bar{v}_{in} + \bar{v}_{ni})^{-1}. \quad (8.23)
\]

The term \(\mathbf{u} \rho_n = \mathbf{u} \left[ \frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \mathbf{u}) \right]\) may be interpreted as the net rate per unit volume at which momentum is added to the neutral fluid as a result of nonelastic collisions involving other species. In writing the approximate interaction terms on the right-hand side of equation (8.5c) we have accounted only for elastic collisions, and so to be consistent it is necessary in the present theory to omit consideration of the effects of nonelastic collisions. Using the global momentum equation (8.7) to substitute for \(D u / D t\), we obtain [cf. equation (II 13.12)]
\[
\frac{\rho_n U_n}{\rho_i} = \frac{-(\rho_n/\rho)^2}{n_e m_{in} \bar{v}_{in}} \mathbf{J} \times \mathbf{B} + \frac{(\rho_n/\rho)}{n_e m_{in} \bar{v}_{in}} \left[ \frac{\rho_n}{\rho} \nabla \cdot p - \nabla \cdot p_n \right]
\]
\[
- \frac{\rho_n m_e}{\rho m_{in} \bar{v}_{in} \mathbf{J}} e n_e. \quad (8.24a)
\]
To obtain an estimate of the second term on the right-hand side of equation (8.24a), let us assume that the neutral mass fraction is approximately uniform and that the pressure tensors are approximately isotropic. Then

\[
\frac{\rho_n}{\rho} \nabla \cdot p - \nabla \cdot p_n \sim \nabla\left(\frac{\rho_n}{\rho} p - p_n\right) \sim \nabla \left[\frac{\rho_n(2p_e + p_n) - (\rho_i + \rho_n)p_n}{\rho}\right]
\]

\[
= \nabla \frac{\rho_n}{\rho} \left(2 - \frac{m_i}{m_n}\right)p_e \sim \frac{\rho_n}{\rho} \nabla p_e,
\]

and equation (8.24a) can be written in the approximate form

\[
\frac{\rho_n}{\rho_i} \frac{U_n}{\rho_i} \approx -\left(\frac{\rho_n}{\rho}\right)^2 \frac{(J \times B - \nabla p_e) - \rho_n}{\rho} \frac{\sqrt{m_e}}{n_e m_{\text{in}} \bar{v}_{\text{in}}} J
\]

(8.24b)

From the global momentum equation we can argue that \(J \times B\) will be of order \(\nabla p\), and thus in the weakly ionized limit we can neglect \(\nabla p_e\) relative to \(J \times B\). To simplify our results, we shall make this approximation for all degrees of ionization, noting that in the fully ionized limit \((\rho_n/\rho)^2 \to 0\), so that the effects of both \(J \times B\) and \(\nabla p_e\) in equation (8.24b) become negligible.

For the third term on the right-hand side of equation (8.21) we obtain

\[
\frac{m_e \bar{v}_{en}}{e} \rho \frac{U_n}{\rho_i} \approx -\frac{(\rho_n/\rho)}{\rho} \frac{\sqrt{m_e}}{n_e m_{\text{in}} \bar{v}_{\text{in}}} \frac{J \times B}{e} - \frac{\sqrt{m_e \bar{v}_{en}}}{\sqrt{n_e m_{\text{in}} \bar{v}_{eh}}} \frac{J}{e}
\]

These terms can be neglected compared to the first two terms on the right-hand side. For the last term in equation (8.21) we obtain

\[
- \frac{\rho_n}{\rho_i} \frac{U_n}{\rho_i} \approx \frac{(\rho_n/\rho)^2}{n_e m_{\text{in}} \bar{v}_{\text{in}}} B \times (J \times B) - \rho_n \frac{\sqrt{m_e}}{\rho} \frac{J \times B}{n_e m_{\text{in}} \bar{v}_{en}}
\]

and thus the only term that we need to retain is

\[
\left(\frac{\rho_n}{\rho}\right)^2 B^2 \frac{b \times (J \times b)}{n_e m_{\text{in}} \bar{v}_{\text{in}}} = \frac{(\rho_n/\rho)^2 \beta_e \beta_i b \times (J \times b)}{\sigma} = \frac{s}{\sigma} b \times (J \times b).
\]

(8.25a)

Here \(\sigma = n_e e^2/m_e \bar{v}_{eh}, \beta_e = eB/m_e \bar{v}_{eh}, \beta_i = eB/m_{\text{in}} \bar{v}_{\text{in}}\), and

\[
\frac{s}{\sigma} = (\rho_n/\rho)^2 \beta_e \beta_i
\]

(8.25b)

is the ion slip factor [cf. equation (3.15)]. Thus, for the conditions (8.11), (8.18), (8.19), (8.20b), and (8.23), the generalized Ohm's law for a partially ionized gas becomes [cf. equations (3.21)]

\[
E' + \frac{\nabla p_e}{e \bar{v}_{en}} = \frac{J + \beta_e J \times b + s b \times (J \times b)}{\sigma}.
\]

(8.26)
This form of the Ohm's law applies to most conditions of interest involving collision-dominated plasmas.

In general, the rate at which electrical energy is dissipated per unit volume in a plasma is given by equation (4.5b) as \( \mathcal{P}_d = \mathbf{J} \cdot \mathbf{E}' \). For a weakly ionized plasma described by the preceding Ohm's law (where the \( \nabla p_e \) term can be neglected), we obtain

\[
\mathcal{P}_d = \frac{J^2}{\sigma} \left( 1 + s \frac{J^2_1}{J^2} \right). \tag{8.27}
\]

Here \( J^2_1 = J^2 - (\mathbf{J} \cdot \mathbf{b})^2 \) is the square of the component of the current density perpendicular to the magnetic induction field \( \mathbf{B} \). Thus, for a given current density the power dissipated locally per unit volume is not affected by the Hall parameter, but is increased by ion slip.

**Exercise 8.1.** Derive the relation (8.9).

**Exercise 8.2.** Derive the relations (8.17) and (8.22).

**Exercise 8.3.** The hydrodynamic treatment of plasmas discussed in this section can be used to derive approximate expressions for the particle diffusion velocities. For the case of no \( B \) field and employing the condition \( (m_e/m_i)^{1/2} (n_e/n_i) \ll 1 \) in place of the charge-neutrality condition (8.11), but retaining the other conditions used in this section, show that

\[
U_e = -\mu_e \left[ \mathbf{E}' + \left( \nabla p_e \right)/en_e \right],
\]

\[
U_i = \frac{\rho_e}{\rho} \mu_e \left( \mathbf{E}' + \frac{\nabla p_e}{en_e} \right) + \frac{\rho_i}{\rho} \mu_i \left( \mathbf{E}' - \frac{V_{pe}}{en_i} \right) + \frac{\rho_i}{\rho} \frac{V_p}{en_i}.
\]

Here \( \mu_e = e/m_e \bar{v}_{eH} \) and \( \mu_i = e/m_i \bar{v}_{iH} \) are the electron and ion mobilities defined in Sec. II 13. Note that in the weakly ionized limit and for constant total pressure, \( U_i = \mu_i \left[ \mathbf{E}' + \left( \nabla p_i \right)/en_i \right] \). [Since the diffusion velocities are often required for situations where \( n_e \) and \( n_i \) can differ, it is convenient to do the calculation by the method used in Sec. II 13, employing equations (8.5a), (8.5b), (8.7), and (8.8).]

**Exercise 8.4.** Calculate the value of the ion slip factor for the conditions specified in Exercise (II 8.2). Assume that the cross section for a potassium ion collision with a particle in the combustion products gas is \( 10^{-14} \) cm\(^2\), and that \( B = 2.7 \) T. What would the number density of neutral particles have to be in order that the ion slip and Hall factors be equal? What would be the value of the Hall parameter for this condition?