7. HARTMANN FLOW

In this section we shall discuss a type of MHD flow that corresponds to a generalization of plane Poiseuille flow in classical fluid mechanics. As illustrated in Fig. 12, we wish to consider the steady and fully developed laminar flow of a conducting fluid in a high aspect ratio rectangular duct, with a uniform and constant magnetic induction field $\mathbf{B}$ oriented normal to the long sides of the duct. We shall refer to the long sides as the insulator walls, and to the short sides as the electrode walls. The distance between the insulator walls is $2a$. The flow between the insulator walls
will be calculated approximately by considering the limiting case where the electrode walls recede to infinity. This type of MHD flow was first investigated by Hartmann (1937). In the absence of a $B$ field and with the requirement that the fluid velocity be zero at the insulator walls, one obtains the well-known result that the velocity profile is parabolic.

Let us orient a reference system with the x-axis along the flow direction and the z-axis in the direction of the $B$ field. We may anticipate that the fluid velocity will have the form $u = [u(z), 0, 0]$ and that all the variables except pressure will depend only on $z$. The motion of the conducting fluid transverse to the $B$ field results in an induced electric field in the negative $y$-direction, $-uB$, and causes a current to flow

$$J_y = \sigma (E_y - uB) \tag{7.1}$$

in accordance with the Ohm's law (6.10d).

From equation (6.10f), $\partial E_y / \partial z = 0$, and hence it follows that the electric field $E_y$ is a constant. The value of $E_y$ depends on the boundary conditions that are imposed by the external circuit at the electrode walls. Electrical energy can either be extracted from or fed into the flowing fluid, depending on whether the opposed electrodes are connected through an external resistance or external voltage supply. In the first case the system acts as a generator, while in the second case it acts as a pump or accelerator. Although there is no difficulty in analyzing these cases (see Sutton and Sherman, 1965, p. 341; or Hughes and Young, 1966, p. 167), we shall, for simplicity, limit our discussion to the open circuit case, defined by the condition

$$\int_{-a}^{a} J_y(z) \, dz = 0. \tag{7.2}$$

Denoting an average with respect to $z$ by a bar, it follows from equations (7.1) and (7.2) that the open circuit electric field is

$$E_y = \frac{\bar{u} \sigma}{\sigma} B. \tag{7.3}$$

In the vicinity of the center of the channel where $u(z)$ is largest $uB$ will exceed $E_y$, and $J_y$ will be negative. Near the insulator walls $u(z) \to 0$, and $J_y$ will be positive. As a result we will have currents flowing in closed loops, as shown in Fig. 12.

The effect of the Hartmann currents on the velocity profile of the fluid can be understood qualitatively in terms of the $(J \times B)_x = J_y B$ force. In the central part of the channel this force is negative and acts so as to brake the fluid. Near the insulator walls this force is positive and therefore
tends to accelerate the fluid. The net result of this interaction is to produce a velocity profile more rectangular in shape than a parabola, flat in the center of the channel, and dropping rapidly at the insulator walls (see Fig. 13). For a given mass flow, we can expect the larger shear stress of the Hartmann flow at the walls to result in a larger pressure gradient in the flow direction.

The $J_y$ current also gives rise to an induced magnetic induction in the $x$-direction, $B_x$. This field may be determined from equation (6.10e) in the form

$$\frac{dB_x}{dz} = \mu_0 J_y,$$  \hspace{1cm} (7.4)

together with the symmetry condition that at the center of the channel, $B_x = 0$. For flows with small magnetic Reynolds numbers, this induced field can usually be neglected.

If we assume the flow is incompressible, then equation (6.10a) is automatically satisfied, and from equation (6.10b) we obtain

$$\frac{\partial p}{\partial x} = J_y B + \frac{d}{dz} \left( \eta \frac{du}{dz} \right),$$ \hspace{1cm} (7.5a)

$$\frac{\partial p}{\partial z} = -J_y B_x.$$ \hspace{1cm} (7.5b)

Since the right hand sides of these equations are functions of $z$, it follows that the pressure must have the form

$$p = p_0 + p_x x + p'(z),$$ \hspace{1cm} (7.6)

where $p_0$ and $p_x$ are constants. The term $p_x x$ drives the flow; the term $p'(z)$ balances the $J \times B$ force in the $z$-direction, and it can be found from equation (7.5b). Combining equation (7.1) with (7.5a), the velocity profile $u(z)$ is determined by the equation

$$\frac{d^2 u}{dz^2} - \frac{\sigma B^2}{\eta} u = -\left( \frac{\sigma B_x - p_x}{\eta} \right).$$ \hspace{1cm} (7.7)

We have assumed here that the viscosity is uniform.

To obtain a closed-form solution we shall assume, following Hartmann, that the electrical conductivity is uniform. This assumption, which makes equation (7.7) linear, is good for liquid metals but, as we shall see, is not very satisfactory for plasmas. The general solution of equation (7.7) is

$$u(z) = C_1 \cosh \left( \frac{Hz}{a} \right) + C_2 \sinh \left( \frac{Hz}{a} \right) + \left( \frac{\sigma B_x - p_x}{\sigma B^2} \right).$$
Here $C_1$ and $C_2$ are constants of integration, and

$$H \equiv aB(\sigma/\eta)^{1/2} \quad (7.8)$$

is the Hartmann number for this problem. Imposing the conditions $u(a) = u(-a) = 0$, we obtain for the velocity profile

$$u(z) = \left(\frac{\sigma B E_y - p_x}{\sigma B^2}\right) \left(1 - \tanh(Hz/a)\right). \quad (7.9a)$$

This result can be written in dimensionless form as

$$\frac{u(z)}{\bar{u}} = H \left(\frac{\cosh H - \cosh(Hz/a)}{H \cosh H - \sinh H}\right), \quad (7.9b)$$

where the average velocity

$$\bar{u} = \left(\frac{\sigma B E_y - p_x}{\sigma B^2}\right) \left(1 - \frac{\sinh H}{H \cosh H}\right) \quad (7.10)$$

is obtained directly from equation (7.9a). The corresponding current density profiles can be written

$$\frac{J_y(z)}{\sigma \bar{u} B} = 1 - \frac{u(z)}{\bar{u}}. \quad (7.11)$$

Several velocity profiles calculated from equation (7.9b) are shown in Fig. 13 and illustrate the expected flattening effect of increasing $H$.

Most experimental studies of Hartmann flow have focused on measuring the effect of $B$ on the shear stress $\tau_w \equiv [\eta \, du/dz]_w$ at the insulator wall. The shear stress may be related to the pressure gradient $p_x$ by integrating equation (7.5a) from $z = -a$ to $z = 0$. For the open circuit condition (7.2), noting that $J_y(z)$ is symmetric about $z = 0$, we obtain

$$\tau_w = -ap_x. \quad (7.12)$$

For the case of uniform electrical conductivity $E_y = \bar{u} B$ and it then follows from equation (7.10) that

$$-p_x = \sigma \bar{u} B^2 \frac{\tanh H}{H - \tanh H}. \quad (7.13a)$$

In the absence of a $B$ field, expanding the right-hand side of (7.13a) for small $H$, we obtain

$$-p_x(B = 0) = \sigma \bar{u} B^2 \frac{3}{H^2} = \frac{3\eta \bar{u}}{a^2}. \quad (7.13b)$$
The relative increase in the wall shear stress resulting from the $B$ field is therefore given by the expression

$$\frac{\tau_w - \tau_w(B = 0)}{\tau_w(B = 0)} = \frac{H^2 \tanh H}{3(H - \tanh H)} - 1. \quad (7.14)$$

The solid curve in Fig. 14 shows the dependence of the relative increase in wall shear stress on Hartmann number predicted by equation (7.14). Measurements made by Hartmann and Lazarus (1937), and by others, on laminar flows of mercury are in good agreement with this theory. The dashed lines in Fig. 14 show measurements made by Sonju and Kruger (1969) on a laminar subsonic flow of an atmospheric-pressure, potassium-seeded, combustion products plasma. The parameter $T_w^*$ refers to the ratio of the wall temperature to the mixed mean temperature of the plasma. Lowering the wall temperature causes a marked decrease in the effect of the $B$ field on the wall shear stress. The explanation for this behavior is that the decreased electrical conductivity of the plasma adjacent to the insulator wall
Figure 14. Effect of the Hartmann number on the relative increase in shear stress. The dashed lines show the measurements of Sonju and Kruger (1969) for a seeded combustion products plasma with various values of the ratio of wall temperature to mixed mean temperature ($T_w = 2620\,\text{K}$). The Reynolds number for the experimental points (based on the hydraulic diameter of the duct) is 1200.

It should be noted that for turbulent flows the effect of the $B$ field on the wall shear stress can be strikingly different compared to laminar flows. Under some conditions the wall shear stress of turbulent flows is actually reduced at moderate values of the Hartmann number. The explanation for this phenomenon is that the $B$ field tends to damp turbulent motion and thereby can act so as to decrease the wall shear stress. Eddy currents induced in turbulent fluid particles crossing lines of magnetic induction interact with the $B$ field so as to oppose the turbulent motions. The Hartmann currents and the turbulent eddy currents interacting with the $B$ field therefore reduce the magnitude of the Hartmann eddy currents, and thus there is less interaction with the $B$ field. For wall temperatures less than about one half the mixed mean temperature, the effect of the $B$ field on the wall shear stress is negligible.
Section 7  Hartmann Flow  205

contribute opposing effects in the determination of the wall shear stress.

In assessing the effects of nonuniform electrical conductivity on the Hartmann currents in plasmas one needs to distinguish between whether the gas is in chemical equilibrium or whether nonequilibrium conditions can prevail. The combustion products plasma studied by Sonju and Kruger (1969) is an example of an equilibrium plasma for which the electrical conductivity is tied directly to the heavy particle temperature. The reduced wall temperature therefore results in a reduced electrical conductivity, which in turn reduces the Hartmann current. On the other hand, for noble gas plasmas for example, the electron temperature and number density are only very weakly coupled to the heavy particle temperature. Measurements of these quantities have been made by Brown and Mitchner (1971) in the laminar boundary layer of an atmospheric-pressure potassium-seeded argon plasma flowing over a cooled surface. These measurements show that, even in the absence of electric currents, both the electron temperature and number density can be considerably elevated relative to their equilibrium values. The Joule heating that would result from the presence of a Hartmann current in the boundary layer would lead to a further increase in these quantities, and therefore also to an increased electrical conductivity. It is quite possible under nonequilibrium conditions for the electrical conductivity in the boundary layer to exceed the core value, even though the walls are cooled. The magnitude of Hartmann flow effects in nonequilibrium plasmas can thus exceed those predicted on the basis of an assumed uniform electrical conductivity. Evidence of this behavior is indicated in the high heat transfer rates to an insulator wall measured by Reilly and Oates (1967) and by Carlson (1969).

The solution of the equations of motion for laminar Hartmann flows in fluids with constant properties, including the Hall parameter in the Ohm's law, can be worked out in closed form, and is discussed by Sutton and Sherman (1965, p. 370). The qualitative features of the effects of the Hall parameter can be explained rather simply. Because the transverse conductivity, as described by equation (3.17a), is decreased by the Hall parameter, the Hartmann current in the y-direction that we have been discussing is reduced; the velocity \( u(z) \) therefore tends to return to a Poiseuille-like profile, and the wall shear stress is reduced. In addition, a Hall current \( J_x(z) \) can now flow in the x-direction. This current interacts with the \( B \) field and results in a cross flow of fluid \( v(z) \) in the y-direction. The Hall current will also give rise to an induced magnetic induction \( B_y(z) \) in the y-direction.

**Exercise 7.1.** Obtain an expression for the total Hartmann current flowing in one direction in a fluid of uniform electrical conductivity.
Calculate the magnitude of this current (in amperes) for the case where the distance between the walls is 5 cm, the viscosity is $7.6 \times 10^{-5}$ kg m$^{-1}$ sec$^{-1}$, the electrical conductivity is 8 mhos m$^{-1}$, the magnetic induction is 2.6 T, and the average fluid speed is 500 m sec$^{-1}$. 