2. PARTICLE MOTION IN UNIFORM AND CONSTANT FIELDS

Let us consider first the motion of a particle of mass \( m \) in a uniform and constant electric field \( \mathbf{E} \). For this case the equation of motion is

\[
m \frac{d\mathbf{w}}{dt} = q\mathbf{E}.
\]

(2.1)

In the absence of collisions, the particle therefore accelerates indefinitely with the acceleration

\[
\frac{q\mathbf{E}}{m}.
\]

(2.2)

Let us consider next the motion of a particle in a uniform and constant magnetic induction field \( \mathbf{B} \). For this case, the equation of motion is

\[
m \frac{d\mathbf{w}}{dt} = q\mathbf{w} \times \mathbf{B}.
\]

(2.3)

If we take the dot product of this equation with \( \mathbf{w} \), we obtain

\[
m \frac{d}{dt} \frac{d\mathbf{w}^2}{2} = q\mathbf{w} \cdot (\mathbf{w} \times \mathbf{B}) = q(\mathbf{w} \times \mathbf{w}) \cdot \mathbf{B} = 0.
\]

Thus, the kinetic energy of the particle \( \epsilon = \frac{1}{2}m\mathbf{w}^2 \) is a constant of the motion. The field \( \mathbf{B} \) can do no work on the particle, as is apparent from the fact that the force \( \mathbf{w} \times \mathbf{B} \) is perpendicular to \( \mathbf{w} \).

To describe the trajectory of the particle, it is convenient to resolve \( \mathbf{w} \) into components parallel and perpendicular to the direction of \( \mathbf{B} \), i.e., \( \mathbf{w} = \mathbf{w}_|| + \mathbf{w}_\perp = x_|| + \mathbf{x}_\perp \). The corresponding component equations of motion become

\[
\frac{d\mathbf{w}_||}{dt} = 0,
\]

(2.4a)

\[
\frac{d\mathbf{w}_\perp}{dt} = \frac{q}{m} \mathbf{w}_\perp \times \mathbf{B}.
\]

(2.4b)

From equation (2.4a), we may conclude that the velocity of the particle in the direction of the magnetic induction is constant. In addition, since \( \epsilon = \frac{1}{2}m(\mathbf{w}_||^2 + \mathbf{w}_\perp^2) \), it follows that \( \mathbf{w}_\perp = |\mathbf{w}_\perp| \) is constant.
If we orient a reference frame with the z-axis in the direction of B as shown in Fig. 2, and denote the components of \( w_\perp \) by \( w_x \) and \( w_y \), equation (2.4b) yields the two equations,

\[
\frac{dw_x}{dt} = \frac{qB}{m} w_y, \quad (2.5a)
\]

\[
\frac{dw_y}{dt} = \frac{-qB}{m} w_x. \quad (2.5b)
\]

Integrating these equations we obtain

\[
w_x = w_\perp \cos (\omega t + \alpha) = dx/dt, \quad (2.6a)
\]

\[
w_y = \mp w_\perp \sin (\omega t + \alpha) = dy/dt, \quad (2.6b)
\]

where \( \alpha \) is a phase angle determined by the initial conditions and where \( w_\perp^2 = w_x^2 + w_y^2 \). The upper sign applies when \( q > 0 \) and the lower when \( q < 0 \). The quantity

\[
\omega \equiv \frac{|q|B}{m} \quad (2.7)
\]
is the *cyclotron* (or Larmor) frequency of the motion. If we integrate equations (2.6), we obtain for the trajectory in the \((x, y)\) plane

\[
(x - x_G) = r \sin (\omega t + \alpha), \quad (2.8a)
\]

\[
(y - y_G) = \pm r \cos (\omega t + \alpha). \quad (2.8b)
\]

The projection of the particle motion in a plane perpendicular to \(B\) is therefore a circle with center \((x_G, y_G)\) and with radius (the Larmor radius)

\[
r = \frac{w_\perp}{\omega}. \quad (2.9)
\]

The sense of rotation in Fig. 2 is clockwise for a positive charge and counterclockwise for a negative charge. Since the *guiding center* \((x_G, y_G)\) moves with a constant speed \(|w_\parallel| = w_z\) along \(B\), the particle trajectory in space will be a *helix* tied to a magnetic field line, as indicated in Fig. 3.

As a numerical example, for an electron with a thermal speed \(w_\perp \approx 10^5\, \text{m/sec}\), moving in a magnetic induction field of 1 Wb/m² (10,000 G), \(\omega = \omega_e = 1.8 \times 10^{11}\, \text{rad sec}^{-1}\) and \(r = r_e = 5.7 \times 10^{-7}\, \text{m}\). These values are of the same order of magnitude as those obtained in Exercise II 8.2 for MHD generator conditions for the electron average momentum transfer collision frequency and mean free path, \(\nu_{eH} = 2.6 \times 10^{11}\, \text{sec}^{-1}\) and \(l_e = 1.2 \times 10^{-6}\, \text{m}\). Since \(\omega_e/\nu_{eH} = 0.7\) for these conditions, on the average the electrons will complete a significant fraction of a circular orbit before being interrupted by collisions.
The important plasma property

$$\beta_e \equiv \omega_e \sqrt{\nu_{eH}}$$

(2.10)

will appear again in our later discussions and is called the electron Hall parameter. Each charged species present in the plasma will, of course, have its own value of $\beta$. As in the preceding example, the value of the Hall parameter for each species provides a measure of the average number of revolutions the corresponding charged particle makes between collisions.

The circular motion of a charged particle induced by an applied magnetic field may be viewed as constituting a small current loop. The direction of the current relative to the applied $B$ field, as given by equations (2.8), is shown in Fig. 4. The current loop itself generates an induced $B$ field

![Figure 4. Current loop and equivalent magnetic dipole induced by an applied $B$ field.](image)

1 Electron-electron collisions also serve to interrupt the gyrations of a particular electron, but by definition these are not included in the average electron heavy particle collision frequency $\bar{\nu}_{eH}$ [see equation (II 13.8)]. However $\bar{\nu}_{eH}$ does include the electron-ion collision frequency $\bar{\nu}_{ei}$, which is either equal to (for the ion charge number $Z$ equal to unity) or greater than the electron-electron collision frequency $\bar{\nu}_{ee}$ [see discussion preceding equation (II 8.9)]. Omission of the term $\bar{\nu}_{ee}$ therefore affects the interpretation of $\beta_e$ by at most a factor of two.
approximately equivalent to that produced by a magnetic dipole. It may be shown in general (see Panofsky and Phillips, 1955, p. 119) that the magnitude of the equivalent magnetic dipole moment $d_m$ of a current loop is given by the product of the current $I$ and the area of the loop $\Sigma$. In the present case $I = |q|/\omega/2\pi$ and $\Sigma = \pi r^2$. Employing the relations (2.7) and (2.9) and noting that the induced magnetic dipole is oriented antiparallel to the applied $B$ field, we obtain the result

$$d_m = \frac{-mw^2/2}{B} b,$$

where $b = B/B$ is a unit vector in the direction of $B$. Materials exhibiting this behavior with respect to the direction of the induced dipole moment are called *diamagnetic* (in contrast to paramagnetic).

For slow variations of $B$ in space and time, it may be shown (see Spitzer, 1962, pp. 9-13) that the diamagnetic moment $d_m$ of a charged particle is nearly constant. A quantity exhibiting such behavior is called an *adiabatic invariant*. An important consequence of the invariance of $d_m$ is that a charged particle moving into a region of increasing $B$ will be reflected. From equation (2.11), an increase in $B$ must be accompanied by an increase in the kinetic energy of the particle associated with motion transverse to $B$. This increased transverse kinetic energy must be accompanied by a decrease in the kinetic energy associated with the motion parallel to $B$, since the total kinetic energy is conserved. Thus $E_\parallel$ will decrease to zero, the direction of motion of the particle parallel to $B$ will be reversed, and the particle will then move towards the region of decreasing $B$. Such a region therefore acts as a *magnetic mirror*. In the earth's magnetic field above the atmosphere, this mechanism plays an important role in maintaining trapped ions in the Van Allen radiation belts. In controlled fusion research, two opposed magnetic mirrors are used in some devices as a method for containing very high-temperature collisionless plasmas.

In the treatment of solid materials, it is usual to separate the total current density $j$ appearing in the Ampère-Maxwell equation (see Sec. VI.2)

$$\nabla \times B = \mu_0 j + \mu_0 \frac{\partial D}{\partial t},$$

into a part $j_f$ associated with free charges and into a part $j_b$ associated with the current loops or equivalent magnetic dipoles of bound charges. It may be shown that $j_b = \nabla \times M$, where $M$ is the magnetic dipole moment per unit volume. With this decomposition, the Ampère-Maxwell equation may be written

$$\nabla \times H = j_f + \frac{\partial D}{\partial t},$$
where the magnetic field intensity $\mathbf{H}$ in the material is defined by the relation $\mu_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}$, or equivalently,

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}).$$

This decomposition is quite general, and is particularly useful for ideally permeable materials which satisfy the relation

$$\mathbf{M} = \chi_m \mathbf{H}.$$

Here $\chi_m$ is a constant of proportionality called the magnetic susceptibility. With this relation we can write $\mathbf{B} = \mu \mathbf{H}$, where the constant $\mu = \mu_0 (1 + \chi_m)$ is the absolute permeability of the material. From equation (2.11) it is apparent that for plasmas $M \propto B^{-1}$. For this reason in the treatment of plasmas there is little to be gained in making a distinction between free and bound charges. It is more convenient to work with the total $\mathbf{j}$, and to use the relation $\mathbf{B} = \mu_0 \mathbf{H}$.

Returning to the problem of particle trajectories, let us consider now the motion of a particle subjected to the two fields $\mathbf{E}$ and $\mathbf{B}$ acting simultaneously, both fields being uniform and constant. For this case, the equations of motion are

$$\frac{dw_\parallel}{dt} = \frac{q}{m} \mathbf{E}_\parallel, \quad \text{(2.12a)}$$

$$\frac{dw_\perp}{dt} = \frac{q}{m} (\mathbf{E}_\perp + \mathbf{w}_\perp \times \mathbf{B}). \quad \text{(2.12b)}$$

From equation (2.12a), the charged particle has a constant acceleration along the direction of $\mathbf{B}$. In the plane perpendicular to $\mathbf{B}$ the qualitative features of the trajectory may be determined by superimposing the separate effects of $\mathbf{B}$ and $\mathbf{E}_\perp$.

Let us consider first the case of a positively charged particle. As illustrated in Fig. 5, with the direction of $\mathbf{B}$ out of the plane of the page, and with $\mathbf{E}_\perp = 0$, the particle rotates in a clockwise sense. With the field $\mathbf{E}_\perp \neq 0$, the particle will be accelerated in the direction of $\mathbf{E}_\perp$ and will therefore have a larger value of $w_\perp$ at the top of its trajectory than at the bottom. Correspondingly, the local value of the Larmor radius at the top of the orbit will be larger than at the bottom, and so the positive particle will drift on the average to the right, as shown. For a negatively charged particle, the initial motion with $\mathbf{E}_\perp = 0$ is counterclockwise. When $\mathbf{E}_\perp \neq 0$, the particle is decelerated in the direction of $\mathbf{E}_\perp$ so that the local value of the Larmor radius at the top of the trajectory is less than at the bottom. However, because of the initial counterclockwise motion, the negative particle will also drift to the right, as shown.
From the preceding argument, one is led to seek a transformation to a new reference frame having some as yet unspecified drift velocity $u_D$, with respect to which the charged particle motion is described by a simpler equation. Since $E \times B$ is a vector in the direction of $u_D$, let us write

$$u_D = \alpha (E \times B),$$

where $\alpha$ is a constant that remains to be determined. If we denote the velocity of the charged particle with respect to the new reference frame by $w'_\perp$, then

$$w'_\perp = w_{\perp} + \alpha E \times B.$$  \hspace{1cm} (2.13)

The equation of motion for $w'_\perp$ is obtained by substituting (2.13) into equation (2.12b), whereupon one obtains

$$\frac{dw'_\perp}{dt} = \frac{q}{m} \left[ E_{\perp} + w'_\perp \times B + \alpha (E \times B) \times B \right]$$

$$= \frac{q}{m} \left[ E_{\perp} + w'_\perp \times B - \alpha (B^2 E - B^2 E_{\parallel}) \right]$$

$$= \frac{q}{m} \left[ (1 - \alpha B^2)E_{\perp} + w'_\perp \times B \right].$$  \hspace{1cm} (2.14)

If one selects $\alpha = 1/B^2$, then

$$u_D = \frac{E \times B}{B^2},$$  \hspace{1cm} (2.15)
and equation (2.14) becomes identical in form to equation (2.4b), which we have shown describes a circular trajectory. Accordingly, the trajectory defined by equation (2.12b) consists of motion in a circle about a guiding center having the velocity $u_p$.\(^2\) A trajectory of this kind would be executed, for example, by a point on the spoke of a rolling wheel, with points located at different distances from the hub producing various possible trajectories. It should be noted that the guiding center velocity is independent of the charge and mass of the particle.

Exercise 2.1. Calculate the values of the electron Hall parameter and Larmor radius for the conditions specified in Exercise II 8.2 and with $B = 2.7 \text{ Wb/m}^2$.

Exercise 2.2. Discuss fully the possible trajectories of a charged particle in the case where $E$ and $B$ are constant in space and time.

Exercise 2.3. A proton in perpendicular electric and magnetic induction fields is observed to make $10^3$ revolutions while moving a distance of 1 cm perpendicular to the two fields. If the electric field is 10 V/cm what is the value of the magnetic induction?

Exercise 2.4. A common method of heating an electrode of a thermionic converter is by electron bombardment. A heated filament is placed behind the electrode, and the emitted electrons are then accelerated by an applied voltage $\phi$ so as to bombard the electrode. The region between the filament and the electrode is kept highly evacuated so as to prevent collisions between electrons and residual gas. We may now suppose that such a device is to be operated with a magnetic field $B$ applied parallel to the surface of the electrode. What is the maximum possible value of $B$ for which the bombardment will continue to occur? Assume that $\phi = 2000 \text{ V}$, $s$ (spacing between filament and electrode) = 1 mm, and that the electrons are emitted from the filament with zero velocity.

Exercise 2.5. Discuss qualitatively and then derive the drift velocity of a charged particle in mutually perpendicular magnetic and gravitational fields. Assume the fields are uniform in space and constant in time. What is the current density in a collisionless plasma subjected to these fields?

\(^2\) As $B \to 0$, $|u_p|$ will exceed the velocity of light, and equation (2.15) is no longer valid. Physically, as $B \to 0$, the electric field has a longer and longer time to increase the particle's speed before the direction of the particle with respect to $E_\perp$ is reversed. When the speed of the particle becomes relativistic, the original equation of motion no longer applies (see Landau & Lifshitz, 1959, p. 59). For most conditions of interest, the limitations imposed in order that $|u_p|$ be nonrelativistic are not very severe. As an example, for $B = 0.1 \text{ Wb m}^{-2}$, equation (2.15) is valid provided $E_\perp < 3 \times 10^5 \text{ V cm}^{-1}$. 