

**Exercise 7.2.** Discuss the voltage-current characteristic for a collision-dominated gas contained between two parallel flat plates. Assume the plates are separated by a distance which is large compared with the ion recombination length $l_R$. Sketch the potential distribution between the plates. (The general case of arbitrary separation between the plates is discussed by McKee, 1967.)

### 8. PROPAGATION OF ELECTROMAGNETIC WAVES

The propagation of electromagnetic radiation is governed by Maxwell's equations. These equations are discussed in Sec. 2 of Chapter VI. As shown in Sec. VI 4, a fundamental solution of Maxwell's equations in a vacuum corresponds to the existence of plane, transverse traveling waves which can transport energy. In this section we wish to describe how the propagation of such waves is modified by the presence of a plasma and how one may derive diagnostic information about the plasma from this fact. We shall limit our discussion to the case of a uniform, unbounded, and nonmoving plasma, characterized by an electron number density $n_e$ and average collision frequency $\bar{v}_{eh}$. We shall neglect compressibility effects. In this section, to simplify notation, we shall write $\bar{v} = \bar{v}_{eh}$.

In a plasma, the magnetic induction $B$ and the magnetic intensity $H$ are almost always treated by the same constitutive relation $B = \mu_0 H$ [cf. equation (VI 2.1f)] as in free space. [See the discussion preceding equations (IV 2.12).] To account for the polarizability associated with the bound electrons of the neutral particles and of the ions, one may write in place of equation (VI 2.1e) the relation $D = \varepsilon E$. Taking the time derivative of the Maxwell-Ampere equation (VI 2.1d) and employing Faraday's equation (VI 2.1c), we then obtain the following equation for the electric field:

$$\nabla^2 E - \nabla(\nabla \cdot E) = \mu_0 \frac{\partial j}{\partial t} + \varepsilon \mu_0 \frac{\partial^2 E}{\partial t^2}. \quad (8.1)$$

Since the plasma is assumed stationary, the total current density $j$ is equal to the conduction current density $J$.

The generalized Ohm's law for a partially ionized gas given by equation (IV 8.17) serves to relate $J$ to $E$. For our present purposes we will suppose that there is no applied $B$ field. There is thus no possibility of ion slip, and equation (IV 8.17) can be shown to have the same form as equation (IV 8.9), which was derived for a fully ionized gas. For a plane transverse electromagnetic wave in a vacuum, the magnitude of $B$ is given by equation (VI 4.5b) as $|B| = |E|/c$, where $c = (\varepsilon_0 \mu_0)^{-1/2}$ is the speed of light in a vacuum. The ratio of the force on an electron resulting from $B$ as compared
to \( E \) is then given by equation (IV 1.1) as of order \((w/c)\), where \( w \) is the electron speed. Since our considerations are restricted to nonrelativistic conditions we can neglect the effect of \( B \) in the Ohm's law, and with the assumptions previously noted, we obtain

\[
\frac{1}{\bar{v}} \frac{\partial J}{\partial t} \approx \sigma \bar{E} - J. \tag{8.2}
\]

Here \( \sigma = n_e e^2/m_e \bar{v} \) is the dc electrical conductivity.

Equations (8.1) and (8.2) provide two coupled relations for \( E \) and \( J \). Let us look for transverse plane wave solutions where \( E \) and \( J \) are given, for example, by the real parts of the expressions

\[
E = E_1 \exp i(kx - \omega t), \tag{8.3a}
\]

\[
J = J_1 \exp i(kx - \omega t), \tag{8.3b}
\]

and where \( E_1 \) and \( J_1 \) are constant vectors perpendicular to the direction of propagation (which we have selected as the \( x \)-axis). We assume here that the angular frequency of the wave \( \omega \) is real. The wave number \( k \) may be complex. For such solutions to exist the conditions imposed by equations (8.1) and (8.2), viz.,

\[
k^2 E = i\omega \mu_0 J + \epsilon \mu_0 \omega^2 E, \tag{8.4a}
\]

\[
- \frac{i(\omega/\bar{v})}{\bar{v}} J = \sigma \bar{E} - J, \tag{8.4b}
\]

must be satisfied. Substituting the expression for \( J \) obtained from (8.4b) into (8.4a), we obtain the following dispersion relation:

\[
k^2 \frac{c^2 k^2}{\omega^2} = \left[ 1 + R - \frac{(\omega_p/\omega)^2}{1 + (\bar{v}/\omega)^2} \right] + i \left[ \frac{(\omega_p/\omega)^2(\bar{v}/\omega)}{1 + (\bar{v}/\omega)^2} \right]. \tag{8.5a}
\]

Substituting for \( \sigma \) and writing \( \mu_0 = 1/\epsilon_0 c^2 \), this equation can be expressed in the form

\[
\frac{c^2 k^2}{\omega^2} = \left[ 1 + R - \frac{(\omega_p/\omega)^2}{1 + (\bar{v}/\omega)^2} \right] + i \left[ \frac{(\omega_p/\omega)^2(\bar{v}/\omega)}{1 + (\bar{v}/\omega)^2} \right]. \tag{8.5b}
\]

Here \( R = (\epsilon - \epsilon_0)/\epsilon_0 \) is the contribution of the bound electrons, and \( \omega_p = (n_e e^2/\epsilon_0 m_e)^{1/2} \) is the plasma frequency [cf. equation (5.5a)]. Transverse plane wave solutions will exist for wave numbers which depend on frequency in accordance with equations (8.5).

\[ ^2 \] More accurate forms of the relation (8.4b) are given in equations (VIII 2.26) and (VIII 4.38) based on kinetic theory, and these can be used to obtain a more precise form of the dispersion relation.
Equation (8.5b) defines a complex wave number

\[ k = \beta + i\alpha. \quad (8.6) \]

Substituting this expression into (8.3a), we obtain

\[ \mathbf{E} = E_1e^{-ax}e^{i(\beta x - \omega t)}. \quad (8.7) \]

Thus the real numbers \( \alpha \) and \( \beta \) are referred to as the attenuation and phase constants, respectively. The wavelength is given by the relation \( \beta = 2\pi/\lambda \), and the phase velocity of the wave is \( v_p = \omega/\beta \). By definition, the index of refraction of the plasma is then \( n = c/v_p = c\beta/\omega \). If there were no free electrons, \( k \) would be a real number, and equation (8.5b) would become \( n^2 = 1 + R \). The quantity \( R \), called the refractivity of the medium, can be calculated from data contained in standard handbooks of physical properties, and for gases it is very small compared with unity. For many plasma conditions of interest

\[ R \ll \frac{(\omega_p/\omega)^2}{1 + (\bar{v}/\omega)^2}, \quad (8.8) \]

and the refractivity contributed by the bound electrons can be neglected. The higher the wave frequency, the larger the electron number density needs to be in order that this condition be applicable. In what follows we shall neglect the contribution of \( R \).

The main features implied by the dispersion relation (8.5b) may be exhibited by considering the limiting case \( \bar{v} \ll \omega \). For this case, where collisions are negligible, we obtain

\[ \left( \frac{ck}{\omega} \right)^2 = 1 - \left( \frac{\omega_p}{\omega} \right)^2. \quad (8.9) \]

If

\[ \omega > \omega_p, \quad (8.10) \]

\( k \) is a real number and the wave propagates without attenuation. On the other hand, if \( \omega < \omega_p \), \( k \) is a pure imaginary number, and the solution (8.7) becomes

\[ \mathbf{E} = E_1e^{-ax}e^{-i\omega t}. \]

Such a solution does not correspond to a propagating wave, and on the average the wave does not transport energy. It is called an *evanescent* wave. Electromagnetic waves with frequencies lower than the plasma frequency, incident on a plasma, will be reflected from the surface. Physically, this behavior may be understood on the basis that if the wave frequency is less than \( \omega_p \), then the charges in the plasma have sufficient time to rearrange
Figure 10. Microwave attenuation caused by a uniform plasma slab.

$d = 3.82 \text{ cm}$

$\omega = 7.1 \times 10^{10} \text{ sec}^{-1}$

$n_{ee} = 1.6 \times 10^{12} \text{ cm}^{-3}$
themselves so as to shield the interior of the plasma from the electromagnetic field.

Collisions tend to modify the preceding conclusions to some extent, but qualitatively the situation is not greatly altered. For a wave of given frequency \(\omega\), we may define a critical electron number density \(n_{ec}\) by the equation

\[
\omega^2 \equiv \frac{n_{ec} e^2}{\varepsilon_0 m_e}. \tag{8.11}
\]

In plasmas with \(n_e < n_{ec}\), waves which previously propagated freely are now attenuated by collisions. For plasmas with \(n_e > n_{ec}\), waves which were previously excluded can now propagate, but are usually strongly attenuated (see Fig. 10).

The effects of collisions are described quantitatively by solving equation (8.5b) for \(\alpha\) and \(\beta\). Let us write the right-hand side of equation (8.5b) in the form

\[
K = K_R + iK_I, \tag{8.12a}
\]

where

\[
K_R = 1 - \frac{(\omega_p/\omega)^2}{1 + (\bar{v}/\omega)^2}, \tag{8.12b}
\]

and

\[
K_I = \frac{(\omega_p/\omega)(\bar{v}/\omega)}{1 + (\bar{v}/\omega)^2}, \tag{8.12c}
\]

and let us write for the left-hand side \(ck/\omega = c\beta/\omega + i\alpha/\omega = \mu + i\chi\). The real quantities \(\mu\) and \(\chi\) are then determined by the equations

\[
\mu^2 - \chi^2 = K_R, \tag{8.13a}
\]

\[
2\mu\chi = K_I. \tag{8.13b}
\]

Adding the squares of equations (8.13a) and (8.13b) and then taking the square root of both sides leads to the equation

\[
\mu^2 + \chi^2 = |K|, \tag{8.13c}
\]

where

\[
|K| = \sqrt{K_R^2 + K_I^2}. \tag{8.14}
\]

\(^3\) The quantity \(K\) is often identified as the effective complex dielectric constant for the medium.
Figure 11. Microwave phase shift caused by a uniform plasma slab.

- $d = 3.82 \text{ cm}$
- $\omega = 7.1 \times 10^{10} \text{ sec}^{-1}$
- $n_e = 1.6 \times 10^{12} \text{ cm}^{-3}$
From equations (8.13a) and (8.13c), we then obtain

\[
\frac{c \alpha}{\omega} = \left( \frac{|K| - K_R}{2} \right)^{1/2},
\]  

(8.15a)
and

\[
\frac{c \beta}{\omega} = \left( \frac{|K| + K_R}{2} \right)^{1/2}.
\]  

(8.15b)

The right-hand sides of equations (8.15) depend on the plasma properties \( n_e \) and \( v \). Thus, from measurements of attenuation and phase shift produced in an electromagnetic wave that passes through a plasma, it is possible to infer values for \( n_e \) and \( v \). For a wave of given frequency \( \omega \), the conditions (8.10) and (8.8) set approximate upper and lower limits on the range of values of \( n_e \) that can be measured with this technique. Thus, for example, visible radiation interferometry (which measures phase shifts) may be used for \( n_e \gtrsim 10^{16} \text{ cm}^{-3} \). Microwaves, on the other hand, can be used for \( n_e \) in the approximate range \( 10^9 \text{ cm}^{-3} \leq n_e \leq 10^{14} \text{ cm}^{-3} \). Calculations, using equations (8.15), of the attenuation and phase shift produced in a microwave of frequency \( \omega = 7.1 \times 10^{10} \text{ sec}^{-1} (\lambda \approx 2.7 \text{ cm}) \) by a slab of plasma of thickness \( d = 3.82 \text{ cm} \), are shown in Figs (10) and (11). The attenuation is shown in units of decibels, defined by the relation

\[
\text{dB} = 10 \log_{10} \left| \frac{\mathbf{E}(x = 0)}{\mathbf{E}(x = d)} \right|^2 = 0.868 \alpha d.
\]

The phase shift is given by the expression

\[
\Delta \Phi = \beta d - \beta_0 d = 2\pi (d/\lambda_0)(n - 1),
\]

where \( \beta_0 = 2\pi/\lambda_0 \) is the wave number in a vacuum. A great deal of work has been directed towards the use of microwaves for diagnostic measurements in plasmas. For further information on this subject, the reader is referred to the book by Heald and Wharton (1965).

REFERENCES


