In 1929, Langmuir (1961) introduced the term plasma for a partially ionized gas in which $\lambda_D$ is small compared to other macroscopic lengths of importance (for example, the macroscopic scale of change in electron number density). Under such circumstances one may make the assumption of electrical neutrality, i.e., $n_i \approx n_e$. The word plasma derives from a Greek word meaning "to mold" and was suggested to Langmuir by his observations of the manner in which the positive column of a glow discharge tended to mold itself to the containing tube.

Exercise 2.1. Consider a neutral plasma of charged particle number density $n_e = n_i = 10^{14} \text{ cm}^{-3}$ and temperature $2500^\circ \text{K}$:

1. Suppose that in some manner all the electrons present in a sphere of radium 1 mm were suddenly removed. Calculate the resulting electric field (in volts/m) at the sphere's surface.
2. Calculate the potential difference through which an electron would need to be accelerated in order to acquire a kinetic energy corresponding to the mean thermal energy of an electron in this plasma.
3. What is the maximum fraction of electrons that can be removed from the sphere such that the resulting potential difference between the center and the surface of the sphere shall not exceed the potential difference calculated in part 2?
4. Calculate the Debye length for this plasma.

3. SHEATHS

One of the most important situations in which charge neutrality does not prevail is in the region of a partially ionized gas immediately adjacent to a solid surface. Such regions are referred to as sheaths. The relevant macroscopic scale here being the distance from the surface, we may anticipate that the assumption of charge neutrality will be violated in a region whose extent is of the order of $\lambda_D$. The detailed structure of sheaths may be quite varied, depending on many factors. The discussion of this section will be limited to the simplest of models and is aimed at bringing out the salient features of sheaths most expeditiously.

An important aspect of the sheath problem concerns the disposition of charged particles which strike the solid surface. For many situations where the surface is cooled, it is possible to regard the surface as nonemitting and catalytic. Under such conditions the incident charged particles are either retained on the solid surface or they recombine and are returned to the gas as neutral particles.

Let us consider the special case of a floating electrode suddenly immersed into a stationary plasma. We shall assume, for simplicity, that the electron
and heavy particle temperatures are equal and uniform. Initially, the electron flux to the electrode will considerably exceed that of the ions because of the relation $C_e \gg C_i$ between the electron and ion thermal speeds. In the steady-state condition there can be no net rate of charge accumulation on the electrode. The electrode therefore quickly acquires a negative potential of such a magnitude that the reduced electron flux is exactly balanced by the ion flux. As illustrated in Fig. 2, the two major features of interest for this problem are the magnitude of the floating potential $\phi_0$ and the spatial extent of the potential distribution $\phi(y)$ associated with the sheath. We shall calculate each of these quantities separately.

To determine $\phi_0$, we need to calculate first the flux density of electrons $\Gamma'_e$ striking the electrode. On the average, electrons striking the electrode experience their last collision a distance approximately one mean free path $l_e$ away from the electrode. We shall assume, somewhat arbitrarily, that the electron velocity distribution $f_e(C)$ at this location is Maxwellian. In addition, in order to keep the calculation simple, we shall assume that conditions are such that the sheath thickness is less than $l_e$. The latter assumption enables us to regard the sheath itself as collisionless and means that $n_e \approx n_i = n$ at the location where $f_e(C)$ is Maxwellian. We shall see that this condition for a collisionless, or free-fall, sheath is approximately equivalent to the requirement $\lambda_D \ll l_e$.

With reference to Fig. 3, only those electrons with velocities directed
Towards the surface and with sufficient energy to overcome the potential barrier will reach the electrode. The minimum y-component of velocity required $C_{y0}$ is given by the equation

$$\frac{m_e C_{y0}^2}{2} = -e\phi_0. \quad (3.1)$$

In accordance with equations (II 6.18a) and (II 6.20), the required one-way electron flux density is given by the expression

$$\Gamma_e^\prime = n \int_{C_y > C_{y0}} f_e(C)C_y d^3C = n \left[ \frac{m_e}{2\pi kT} \right]^{1/2} \int_{-\infty}^{\infty} e^{-m_e C_x^2/2kT} dC_x \cdot \left[ \frac{m_e}{2\pi kT} \right]^{1/2} \int_{-\infty}^{\infty} e^{-m_e C_x^2/2kT} dC_x \cdot \left[ \frac{m_e}{2\pi kT} \right]^{1/2} \int_{-\infty}^{\infty} e^{-m_e C_x^2/2kT} dC_x.$$ 

Noting that the brackets containing the $dC_x$ and $dC_y$ integrals have the value unity, we obtain the result

$$\Gamma_e^\prime = \frac{\bar{C}_e}{4} e^{\phi_0/kT}. \quad (3.2)$$

Here $\bar{C}_e = (8kT/m_e)^{1/2}$ is the electron mean thermal speed [cf. equation (II 6.34)]. The one-way ion flux density $\Gamma_i^\prime$ striking the electrode may be calculated in a similar manner, but since the ions are not impeded by the floating potential, one obtains the result

$$\Gamma_i^\prime = \frac{n\bar{C}_i}{4}. \quad (3.3)$$
The floating potential is obtained from the condition
\[ \Gamma_i^+ = \Gamma_e^+ \] (3.4)
For the case of equal electron and ion temperatures, this condition yields the result
\[ -\phi_0 = \frac{kT}{e} \ln \frac{C_e}{C_i} = \frac{kT}{e} \ln \left( \frac{m_i}{m_e} \right)^{1/2} \] (3.5)
We may note that \( \phi_0 \) is proportional to the temperature in eV, as would be expected. However, the factor \( \ln(m_i/m_e)^{1/2} \), which has a minimum value of 3.8 (corresponding to hydrogen ions), provides a significant amplification of the temperature effect.

A more detailed analysis substantiates the result (3.2), but for an infinite flat electrode, equation (3.3) should be changed (see Davison, 1957, p. 116) to read
\[ \Gamma_i^+ \approx \frac{nC_i}{2} \] (3.6)
A large fraction of the electrons moving toward the electrode from a distance one mean free path away are reflected by the potential barrier and return to this location. The electron distribution function is therefore nearly isotropic in velocity space and can be closely approximated by a Maxwellian, thereby justifying the assumption made in obtaining (3.2). On the other hand, all of the ions which leave this location are captured by the electrode so that the ion distribution function is highly nonisotropic in velocity space, and the assumption of a Maxwellian provides a rather crude approximation.

Turning now to the calculation of the potential distribution \( \phi(y) \) in the sheath, we find that treating the ions accurately is made difficult for the aforementioned reason. We may assume that the electrons in the sheath are approximately in thermodynamic equilibrium and thus, in accord with statistical mechanical considerations (see, for example, Tolman, 1938, p. 89), have the Boltzmann distribution
\[ n_e(y) = ne^{-e\phi(y)/kT} \] (3.7)
We have used here the fact that the potential energy of an electron at the position \( y \) is \(-e\phi(y)\). [This result may also be regarded from a dynamical point of view as a statement of conservation of momentum, the electric field force being balanced by a gradient in momentum flux density—see equations (7.4) and (7.6).] If the ions were also in thermodynamic equilibrium, we would have
\[ n_i(y) = ne^{-e\phi(y)/kT} \] (3.8)
This relation would describe the ions well if the electrode reflected all the ions, but it is not a good description for a catalytic surface. However, for simplicity, we shall use the relation (3.8), and we may anticipate that the results will be approximately correct, particularly in view of the fact that equations (3.3) and (3.6) differ only by a factor of two. This statement is supported by more detailed calculations (see Thompson, 1962, p. 29; and Tanenbaum, 1967, p. 210).

The potential distribution for a steady-state problem is governed in general by Poisson's equation

\[
\nabla^2 \phi = -\frac{\rho^e}{\varepsilon_0}.
\]

In terms of \( \phi \), the electric field is given by

\[
E = -\nabla \phi.
\]

The existence of such a potential is a consequence of Faraday's equation for a steady state, \( \nabla \times E = 0 \) [cf. equation (VI 2.1c)]. Poisson's equation results upon substituting for \( E \) in Gauss' equation (2.1).

Employing equations (3.7) and (3.8) in the relation \( \rho^e = e(n_i - n_e) \), the equation for \( \phi(y) \) becomes

\[
\frac{d^2 \phi}{dy^2} = -\frac{ne}{\varepsilon_0} \left( e^{-e\phi(y)/kT} - e^{e\phi(y)/kT} \right).
\]

Towards the plasma edge of the sheath where \( |e\phi/kT| \ll 1 \), the exponentials may be expanded so that

\[
\frac{d^2 \phi}{dy^2} = -\frac{ne}{\varepsilon_0} \left( 1 - \frac{e\phi}{kT} - 1 + \frac{e\phi}{kT} + \cdots \right) \approx \frac{2ne^2}{\varepsilon_0 kT}, \phi = \frac{2}{\lambda_0^2} \phi,
\]

and therefore

\[
\phi(y) \propto e^{-\sqrt{2y}/\lambda_0}.
\]

An exact integration of equation (3.11) subject to the boundary conditions that \( \phi(0) = \phi_0 \) and that as \( y \to \infty, \phi \) and \( d\phi/dy \to 0 \), yields the result

\[
\frac{e\phi}{kT} = \ln \left( \frac{1 - \tanh(e\phi_0/4kT) e^{-\sqrt{2y}/\lambda_0}}{1 + \tanh(e\phi_0/4kT) e^{-\sqrt{2y}/\lambda_0}} \right).
\]

Numerical comparison of equations (3.13) and (3.12) shows that the solution for all \( y \) is well-represented by the relation

\[
\phi(y) = \phi_0 e^{-\sqrt{2y}/\lambda_0}.
\]

Our major conclusion from this result is to observe that the spatial extent of the sheath is of the order of the Debye length, confirming what we had
anticipated. The fact that the plasma attempts to adjust itself near the electrode surface so as to shield the main body of the plasma from the electric field is a general property of plasmas.

**Exercise 3.1.** Calculate the sheath thickness for a plasma adjacent to an isolated material surface when the electrons and ions are in thermal equilibrium at different temperatures $T_e$ and $T_i$. Assume that the sheath thickness is much less than a mean free path.

4. **SHIELDED COULOMB POTENTIAL**

Suppose we place a point charge $q > 0$ in a plasma and inquire as to the steady-state distribution of charge that results in the neighborhood of $q$. As illustrated in Fig. 4, on the average there will be a surplus of negative charge in the immediate vicinity of $q$ which gradually diminishes as the radial distance $r$ from $q$ is increased. This distribution results from the simultaneous tendency of $q$ to attract electrons and repel positive ions. For sufficiently large values of $r$, the plasma will return to a condition of macroscopic electrical neutrality. The result of this redistribution of plasma charge is to produce an electric field at distant points in the plasma which completely cancels the electric field produced by $q$.

The shielded Coulomb potential $\phi(r)$ which describes the net effect of the charge $q$ and the distributed plasma space charge is determined by Poisson's equation (3.9) in the form

$$\nabla^2 \phi = -\frac{e}{\epsilon_0} [n_i(r) - n_e(r)]. \quad (4.1)$$

Since the plasma near $q$ is in thermodynamic equilibrium, we may use relations (3.7) and (3.8), expressed in terms of the radial distance $r$, to