At what temperature would the argon atoms have the same mean speed as the (2500°K) electrons?

**Exercise 6.7.** For a given recombination process, the approximate relation \( a \approx g_e Q^{\text{recomb}} \) may be used to define a corresponding average recombination cross section. Using the data in Table 7, calculate a value for the average cross section for radiative recombination.

**Exercise 6.8.** In accordance with equation (6.18a), the vector differential flux density relative to the mean mass velocity of the fluid for any particle species may be written \( d\Gamma = n_f(C)C d^3C \). For particles with a Maxwellian velocity distribution, show that the total number of particles passing per unit time through a unit area (from one side to the other) is given by the expression

\[
\frac{nC}{4},
\]

where \( \overline{C} = (8kT/\pi m)^{1/2} \).

**Exercise 6.9.** For the case where the type-1 particles have a Maxwellian distribution at a temperature \( T_1 \), and the type-2 particles are Maxwellian at a temperature \( T_2 \), show that instead of equation (6.28), one obtains

\[
R_{12} = n_1 n_2 \left[ \frac{8k}{\pi} \left( \frac{T_1}{m_1} + \frac{T_2}{m_2} \right) \right]^{1/2} \int_0^\infty xe^{-xQ_{12}(g)} dx.
\]

**Exercise 6.10.** The random current density for each species in a gas is defined as the charge carried by the species per unit time through a unit area (from one side to the other). Calculate the random electron and ion currents (in amperes per cm\(^2\)) in a gas containing partially ionized potassium at a temperature of .216 eV with \( n_e = n_i = 10^{13} \text{ cm}^{-3} \). Assume that the particles have Maxwellian velocity distributions.

7. **ENERGY AND MOMENTUM TRANSFER IN ELASTIC COLLISIONS**

In an elastic two-body collision, the total kinetic energy of the particles before and after the collision remains the same. However, in any reference frame other than the center-of-mass frame for the two particles (see Exercise 6.2), the speeds of the two particles will not remain unchanged. For example, a test particle incident on a field particle which is initially stationary will cause the field particle after the collision to exhibit motion. The test particle must therefore have lost some of its initial kinetic energy.

Consider a fluid in motion described by a macroscopic mean mass velocity field \( u = u(x, t) \). Let us consider a collision between two particles
located at position $x$ at time $t$, as viewed from a local reference frame having the instantaneous velocity $u$. On a macroscopic distance scale there is no need to distinguish between the actual positions of the two particles, and on a macroscopic time scale the collision time is infinitesimally small. Let us suppose that the velocities of the two particles in this reference frame before the collision are $C$ and $W$, and that the collision results in a scattering described by the angles $(\chi, \phi)$ in the relative reference frame, as shown in Fig. 9. In accordance with equation (6.23a), the velocities of the test particle before and after the collision may be written

$$C = G + \frac{m_{12}}{m_1} g,$$  \hspace{1cm} (7.1a)$$

$$C' = G + \frac{m_{12}}{m_1} g'. \hspace{1cm} (7.1b)$$

We have employed here the fact that $G' = G$, which follows from the condition of conservation of momentum in an elastic collision (see Exercise 6.2).

The energy lost by the test particle in the particular collision just defined is

$$\Delta \epsilon_{12}(C, W; \chi, \phi) \equiv \frac{m_1 C^2}{2} - \frac{m_1 C'^2}{2} = m_{12} G \cdot (g - g'). \hspace{1cm} (7.2)$$

The energy lost by the test particle averaged over all scattering angles is

$$\Delta \epsilon_{12}(C, W) \equiv \int_{4\pi} \Delta \epsilon_{12}(C, W; \chi, \phi) p_{12}(\chi, \phi) \, d\Omega$$

$$= m_{12} G \cdot \int_{4\pi} (g - g') \frac{L^{(e)}_{12}(\chi)}{Q^{(e)}_{12}} \, d\Omega, \hspace{1cm} (7.3a)$$

where $p_{12}(\chi, \phi) \, d\Omega$ is the probability of elastic scattering into the solid angle $d\Omega$, as defined by equation (3.8). To evaluate the integral appearing in equation (7.3a), take a special coordinate system with the $z$-axis parallel to the direction of $g$. Since $g' = [g \sin \chi \cos \phi, g \sin \chi \sin \phi, g \cos \chi]$, we may write

$$\int_{4\pi} (g - g') L^{(e)}_{12}(\chi) \, d\Omega$$

$$= g \int_{0}^{\pi} L^{(e)}_{12}(\chi) \sin \chi \int_{0}^{2\pi} [-\sin \chi \cos \phi, -\sin \chi \sin \phi, (1 - \cos \chi)] \, d\phi \, d\chi$$

$$= g \int_{0}^{\pi} L^{(e)}_{12}(\chi) \sin \chi [0, 0, 2\pi(1 - \cos \chi)] \, d\chi = g[0, 0, Q^{(l)}_{12}(g)] = Q^{(l)}_{12}(g)g.$$
Employing equations (6.22), it follows that

\[ \Delta \epsilon_{12}(C, W) = m_{12} \frac{Q_{12}^{(1)}(g)}{Q_{12}^{(0)}(g)} G \cdot g \]

\[ = \frac{m_{12}}{(m_1 + m_2)} \frac{Q_{12}^{(1)}(g)}{Q_{12}^{(0)}(g)} (m_1 C + m_2 W) \cdot (C - W). \]  

(7.3b)

Let us consider first the special case where the field particle is initially stationary, i.e., \( W = 0 \). We then obtain from equation (7.3b)

\[ \Delta \epsilon_{12}(C, 0) = \frac{2m_{12}}{(m_1 + m_2)} \frac{Q_{12}^{(1)}(g)}{Q_{12}^{(0)}(g)} \left( \frac{m_1 C^2}{2} \right). \]

Thus, the average fraction of the initial kinetic energy lost by a test particle of speed \( C \) incident on a stationary field particle is

\[ \frac{\Delta \epsilon_{12}(C, 0)}{m_1 C^2/2} = \frac{2m_{12}}{(m_1 + m_2)} \frac{Q_{12}^{(1)}(C)}{Q_{12}^{(0)}(C)}. \]  

(7.4)

This result is most frequently applied to collisions in which the test particle is an electron and the field particle is a heavy particle such as an atom, molecule, or ion. Since \( Q_{12}^{(1)} \simeq Q_{12}^{(0)} \), we obtain

\[ \frac{\Delta \epsilon_{eh}(C, 0)}{m_e C^2/2} \simeq \frac{2m_e}{m_h}, \]  

(7.5)
where $m_h$ denotes the mass of the heavy particle. The fact that the mass ratio on the right-hand side of equation (7.5) is small means that in some situations the energy coupling between electrons and heavy particles in a gas can be quite weak. Under such circumstances it becomes useful to introduce the concept of the electron temperature $T_e$, which may be quite different from the heavy particle temperature $T$. This situation occurs frequently, for example, when one is dealing with discharges in gases, where $T_e > T$ (see Chapter IV, Sec. 5). We may also introduce the energy exchange collision frequency

$$v_{eh}^E \equiv \frac{2m_e}{m_h} v_{eh}^{(1)}. \quad (7.6)$$

The reciprocal of $v_{eh}^E$ provides a measure of the characteristic time in which an electron is able to transfer its energy to heavy particles by means of elastic collisions.

Let us now return to equation (7.3b) and calculate the energy lost in elastic collisions by a test particle of velocity $C$, averaged over the velocities of all field particles of type-2 present in the gas, i.e.,

$$\Delta \epsilon_{12}(C) = \int \Delta \epsilon_{12}(C, W) f_2(W) \, d^3W. \quad (7.7)$$

In order to proceed with this calculation, we shall assume that the ratio $Q_{12}^{(1)}/Q_{12}^{(2)}$ is approximately independent of relative speed. Using the definitions for the diffusion velocity [see equation (6.6)]

$$U_2 = \int W f_2(W) \, d^3W, \quad (7.8)$$

and the kinetic temperature

$$\frac{3}{2} kT_2 = \int \frac{m_2 W^2}{2} f_2(W) \, d^3W = \frac{m_2 \bar{W}^2}{2}, \quad (7.9)$$

we obtain

$$\Delta \epsilon_{12}(C) = \frac{2m_1}{(m_1 + m_2) Q_{12}^{(2)}} \left[ \frac{m_1 C^2}{2} - \frac{3}{2} kT_2 + \frac{(m_2 - m_1)}{2} C \cdot U_2 \right]. \quad (7.10)$$

Finally, we may calculate the average energy lost by an average test particle per collision, as a result of elastic collisions with all type-2 field particles

$$\bar{\Delta \epsilon}_{12} = \int \Delta \epsilon_{12}(C) f_1(C) \, d^3C. \quad (7.11a)$$
We obtain the result

\[ \overline{\Delta \epsilon_{12}} = \frac{2m_2}{(m_1 + m_2)} \frac{Q_{12}^{(1)}}{Q_{12}^{(e)}} \left[ \frac{3}{2} k(T_1 - T_2) + \frac{(m_2 - m_1)}{2} U_1 \cdot U_2 \right], \quad (7.11b) \]

or

\[ \overline{\Delta \epsilon_{12}} = \frac{2m_2}{(m_1 + m_2)} \frac{Q_{12}^{(1)}}{Q_{12}^{(e)}} \left[ (1 - \frac{T_2}{T_1}) + \frac{(m_2 - m_1)}{m_1} \frac{U_1 \cdot U_2}{C^2} \right]. \quad (7.11c) \]

The order of magnitude of the second term may be written

\[ \frac{m_2 - m_1}{\sqrt{m_2 m_1}} \left( \frac{U_1}{C} \right) \left( \frac{U_2}{W} \right). \]

For collision-dominated gases the ratio of the diffusion speed to the thermal speed for any species $U/C$, is usually very small, so that the order of the second term is much less than one.\(^3\) In such cases equation (7.11b) may be written in the approximate form

\[ \overline{\Delta \epsilon_{12}} \simeq \frac{2m_2}{(m_1 + m_2)} \frac{3}{2} k(T_1 - T_2). \quad (7.11d) \]

Equation (7.11d) provides a good estimate of the actual average energy lost per collision by electrons in collisions with atoms, particularly at temperatures below the threshold energy of the first excited level. However, it is not possible to neglect the effects of nonelastic collisions when one is concerned with electron-molecule collisions. To account for such nonelastic collisions, one may introduce the nonelastic energy loss factor $\delta_h$, and write the average energy lost by an average electron per collision as

\[ \overline{\Delta \epsilon_{eh}} = \frac{2m_e}{m_h} \delta_h \frac{3}{2} k(T_e - T). \quad (7.12) \]

Experimental data on $\delta_h$ for several molecular gases at room temperature, obtained for very weakly ionized conditions where the electron distribution function was likely non-Maxwellian, are shown in Fig. 10 (see also Massey, 1969, p. 727). Since there are complexities involved in the reduction of the experimental data to the form exhibited in Fig. 10 (see Massey and Burhop, 1969, p. 53), these results are presented for illustrative purposes only to show that $\delta_h$ can differ from unity by several orders of magnitude.

\(^3\) For essentially the same reason, this second term does not appear in the results of the rigorous theory of Chap. VIII. The reader should be cautioned against accepting uncritically such “extra” terms of usually small magnitude which have been obtained on the basis of an approximate analysis.
It should be noted that this method of accounting for nonelastic energy losses in electron-molecule collisions is approximate even if, as is assumed here, there is a high rate of energy exchange between the rotational and vibrational modes with the translational mode. When the coupling is weak, a more detailed treatment is necessary (see, for example, Bender, Mitchner, and Kruger, 1971).

Using the same approach as that described at the beginning of this section, we may also calculate the average momentum lost in elastic collisions between particles of two species in a gas. From equation (7.1) we have first, for a particular collision

\[ \Delta p_{12}(C, W; \chi, \phi) = m_1 C - m_1 C' = m_{12}(g - g'). \]  

(7.13)
Averaging over scattering angle, we obtain
\[ \Delta p_{12}(C, W) = m_{12} \frac{Q_{12}^{(1)}(g)}{Q_{12}^{(e)}(g)} g = m_{12} \frac{Q_{12}^{(1)}}{Q_{12}^{(e)}} (C - W). \] (7.14)

If we now assume again that \( Q_{12}^{(1)}/Q_{12}^{(e)} \) is approximately independent of relative speed, the momentum lost by a test particle of velocity \( C \) per collision, averaged over collisions with all type-2 field particles is
\[ \Delta p_{12}(C) \equiv \int \Delta p_{12}(C, W)f_2(W) \, d^3W
\]
\[ = m_{12} \frac{Q_{12}^{(1)}}{Q_{12}^{(e)}} (C - U_2). \] (7.15a)

Under most circumstances \( |U_2| \ll |C| \), and so we have the approximate result, which we shall employ in Sec. 12, that
\[ \Delta p_{12}(C) \approx \frac{m_{12}}{m_1} p_1, \] (7.15b)

where \( p_1 = m_1 C \) is the initial momentum of the test particle. Finally, averaging equation (7.15a) over all test particles, we obtain the average momentum lost per elastic collision of test particles with type-2 field particles
\[ \bar{\Delta} p_{12} = m_{12} \frac{Q_{12}^{(1)}}{Q_{12}^{(e)}} (U_1 - U_2)
\]
\[ \approx m_{12}(U_1 - U_2). \] (7.16)

The expressions (7.11d) and (7.16) for the average energy and momentum transfer in elastic collisions will be employed in Chapter IV in formulating a multifluid theory for partially ionized gases.

**Exercise 7.1.** What is the rate of energy loss per unit volume (in watts/cm\(^3\)) from electrons at 3000\(^\circ\)K colliding with helium atoms at 2000\(^\circ\)K? The electron number density is 10\(^{13}\) cm\(^{-3}\) and the total gas pressure is 1 atm. What is the electron temperature equilibration time? If the helium were replaced with carbon dioxide, what would the electron temperature equilibration time be?

**Exercise 7.2.** Helium at a stagnation pressure of 2.90 atm and at a stagnation temperature of 2500\(^\circ\)K is seeded with a small amount of cesium vapor and is expanded isentropically in a supersonic nozzle 10 cm long to a Mach number of three. The temperature of the helium and other heavy particles drops to 625\(^\circ\)K and the helium number density decreases to
10^{18} \text{ cm}^{-3}$. Estimate the downstream electron temperature. Assume the electron and heavy particle temperatures are equal before the expansion.

**Exercise 7.3.** Using the random walk concept embodied in equation (5.9), obtain an approximate expression for the average distance $l_E$ traversed by a "hot" electron before it comes into energy equilibrium with a background gas consisting of heavy particles. What would the value of this "electron energy-equilibration length" be for atmospheric-pressure argon at a temperature of 2000°K?

**Exercise 7.4.** Consider a partially ionized gas mixture containing potassium and argon at a temperature of 2200°K and at a pressure of 1 atm. The mole fraction of potassium is 0.001.

1. What is the approximate "mean free path" a "hot" electron must travel in order to equilibrate its temperature with that of the heavy particles?
2. What mole fraction of CO$_2$ should be added to the gas to reduce the electron temperature equilibration distance to 0.01 cm?

(Suggestion: To simplify the problem, neglect electron collisions with potassium and with potassium ions; also take the molecular weight of CO$_2$ to be approximately the same as that of argon.)

**Exercise 7.5.** In a helium gas, how many elastic collisions on the average does a cesium atom experience before its momentum is appreciably changed?